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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

ELASTIC PROPERTIES OF RIVETED CONNECTIONS

BY J. CHARLES RATHBUN,[1] M. AM. SOC. C. E.

SYNOPSIS

In designing floor-beams for steel-frame buildings, it is customary to neglect the restraining effect of end connections and to assume that the beams are hinged at the ends. In many other problems of stress analysis in steel frames, it is customary to treat the connections as completely fixed, or rigid.

While it is recognized that the end connections do provide some degree of restraint there has been no means of allowing for it because no one has known, numerically, how much restraint there is. One could not safely take advantage of any saving resulting from the resistance due to bending, created by the connections.

Tests of eighteen end connections of the three most common types of steel beams, form the basis of this paper.[2] The writer has shown how these data may be applied in the design, not only of single beams, but also in the investigation of the stresses in frames, composed of beams and columns, joined by these typical riveted connections.

His analysis shows that economics in design can be effected safely in some cases. It also gives the engineer a more complete understanding of the action of standard beam connections, and indicates several problems for further research.

NOTE.—Discussion on this paper will be closed in April, 1935, *Proceedings*.

[1] Associate Prof., Civ. Eng., Coll. of the City of New York, New York, N. Y.

[2] Prepared from a dissertation presented to Columbia University in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Faculty of Pure Science.

Introduction

The purpose of this paper is to record data derived from a series of tests of standard riveted beam connections and to show the method of incorporating this information in the analysis of steel frames. These tests were conducted in the Materials Testing Laboratory of the School of Technology of the College of the City of New York. Some idea is afforded of the rigidity of several types of connections and of the effect of yielding in these connections on the stress distribution in steel structures, such as building frames.

All the so-called "exact methods" of solution of the building frame under loads are based on the assumption (among others) that the horizontal members are rigidly connected to the vertical. This may lead to serious error. Unlike the assumption that the members do not change in length under stress, the effect of yielding in the connections cannot be determined by the analyst alone. Laboratory experiments are necessary in order to establish constants to incorporate into any analysis that may be made.

Little has been published on this important subject; to date, only a few isolated tests have been made on riveted connections for the purpose of obtaining some measure of their capacity to resist moment. In 1917, a series of tests was made at the University of Illinois, Urbana, Ill.[3] C. R. Young, M. Am. Soc. C. E., has conducted tests at the University of Toronto, Toronto, Ont., Canada, primarily for the purpose of comparing riveted, with welded, connections.[4] The writer knows of no other tests, at all extensive, on the subject.

Although it is probable that many of the connections encountered in a wind-stress design of buildings will be larger than those represented in the data herein submitted, the results of this investigation should serve as a guide in evaluating the properties of the larger connections. The capacity of the testing machine limited the size of the specimens tested.

The paper consists of two parts: Part I is a report on a series of physical tests to determine the relationship between the angular change in riveted beam connections and the moment inducing these changes; and Part II shows the necessary changes that must be made in the formulas used in the several current methods of analysis in order to allow for the elasticity of the riveted connections.

PART I.—PHYSICAL TESTS ON RIVETED CONNECTIONS

Test Specimens

Ordinary shop practice was followed in fabricating the specimens used in this work, no special care being taken because of the fact that the specimens were to be subjected to tests.

The physical properties of the steel in the various specimens of these tests are given in Table 1.

[3] *Bulletin No. 104*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

[4] The report of this set of tests was read at the Annual Convention of the American Society of Civil Engineers in Chicago, Ill., in 1933, and pub. in the *Canadian Journal of Research*, July and August, 1934.

TABLE 1.—TESTS OF STEEL

MEMBER		UNIT STRESSES, IN POUNDS PER SQUARE INCH		Percentage elongation	Percentage reduction	CHEMICAL ANALYSIS (PERCENTAGES)	
Symbol	Weight, in pounds per foot	Elastic limit	Tensile strength			Phosphorus	Silicon
H-140....	167	41 200	61 340	30.0	53.1	0.014	0.037
G- 22....	101	43 220	60 540	30.0	61.4	0.014	0.030
G- 16....	83	42 540	60 420	28.7	56.9	0.016	0.032
G- 15....	99	41 950	59 470	28.7	59.1	0.015	0.034
S- 24....	105.9	45 860	64 640	27.5	50.2	0.013	0.031
S- 18....	54.7	44 810	58 980	27.5	54.0	0.016	0.043
S- 8....	18.4	39 840	62 320	26.2	55.6	0.014	0.042
S- 6....	12.5	44 000	64 800	27.5	60.0	0.017	0.032
I- 12....	31.8	47 120	62 900	26.2	58.25	0.018	0.036

DESCRIPTION OF TEST SPECIMENS

Eighteen specimens (see Figs. 1 and 2) representing as many connections were tested for the purpose of furnishing data from which their elastic resistance to moment could be obtained. These specimens may be divided into three types: Series *A*, Specimens 1 to 7 (Fig. 1), are standard clip-angle connections; Series *B*, Specimens 8 to 12 (Fig. 1) are seat-angle connections; and Series *C*, Specimens 13 to 17 (Figs. 1 and 2) are a series of shallow wind-braced connections. Specimen 18 was designed to furnish some information on the effect of the post on the joint. In Series *A* and *B*, $\frac{7}{8}$-in. rivets were driven in $\frac{15}{16}$-in. holes. In Series *C*, except Specimens 13 and 14, 1-in. rivets were driven in $1\frac{1}{16}$-in. holes. The lighter pieces were punched and the heavier ones were drilled.

Each specimen consisted of a central plate with an **I**-beam abutting on either side fastened by means of the standard connection to be investigated. The specimen thus formed was tested by applying a load on the central plate, while it was supported at the outer ends of the **I**-beams. Thus, the specimen acted as a simple beam loaded at the center and a known moment was induced in the two connections.

Series *A* represents a type of connection that is not designed primarily to resist moment, and it is customary to consider these connections as hinged. However, they have some measure of elasticity and it was hoped that these tests would furnish the data necessary to enable a designer to estimate this elasticity in those cases where it seriously affected stress distribution. In designing Series *A*, standard connections were chosen from those published in handbooks issued by steel manufacturers, taking the double-riveted connections and the companion single-riveted connections for the 8-in., 12-in., and 18.-in. **I**-beams. Thus, a study of double riveting as it affected the stiffness of the connections was made. The spread in size from the 8-in. to the 18-in. **I**-beams was as great as was considered practicable. As the 6-in. standard connection is probably the most common, it was added to the series. A comparison of the several connections showing the effect of adding a second row of rivets can be made by studying Fig. 3.

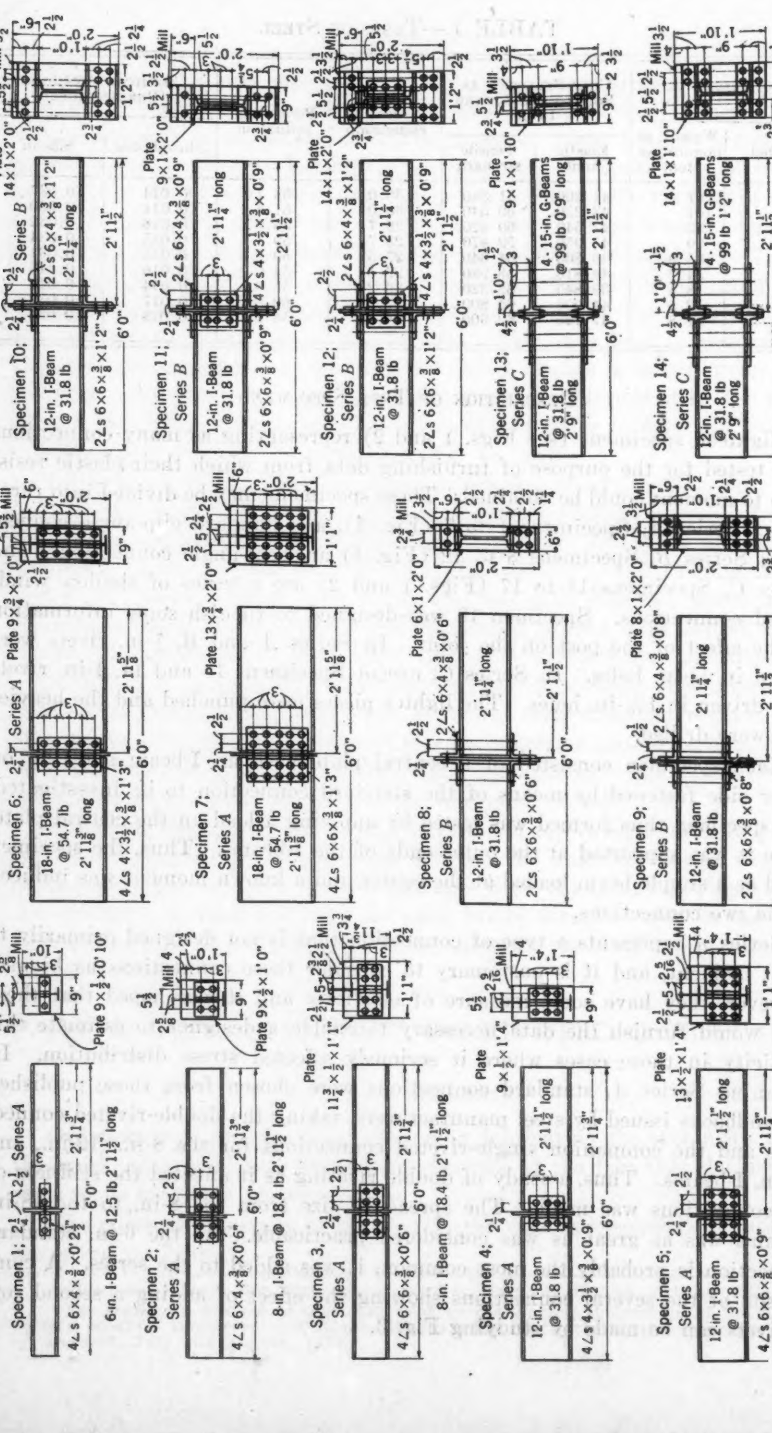


FIG. 1.—TEST SPECIMENS OF RIVETED CONNECTIONS, WITH ⅞-IN. ϕ RIVETS; SERIES *A*, SPECIMENS 1 TO 7; SERIES *B*, SPECIMENS 8 TO 12; AND SERIES *C*, SPECIMENS 13 AND 14.

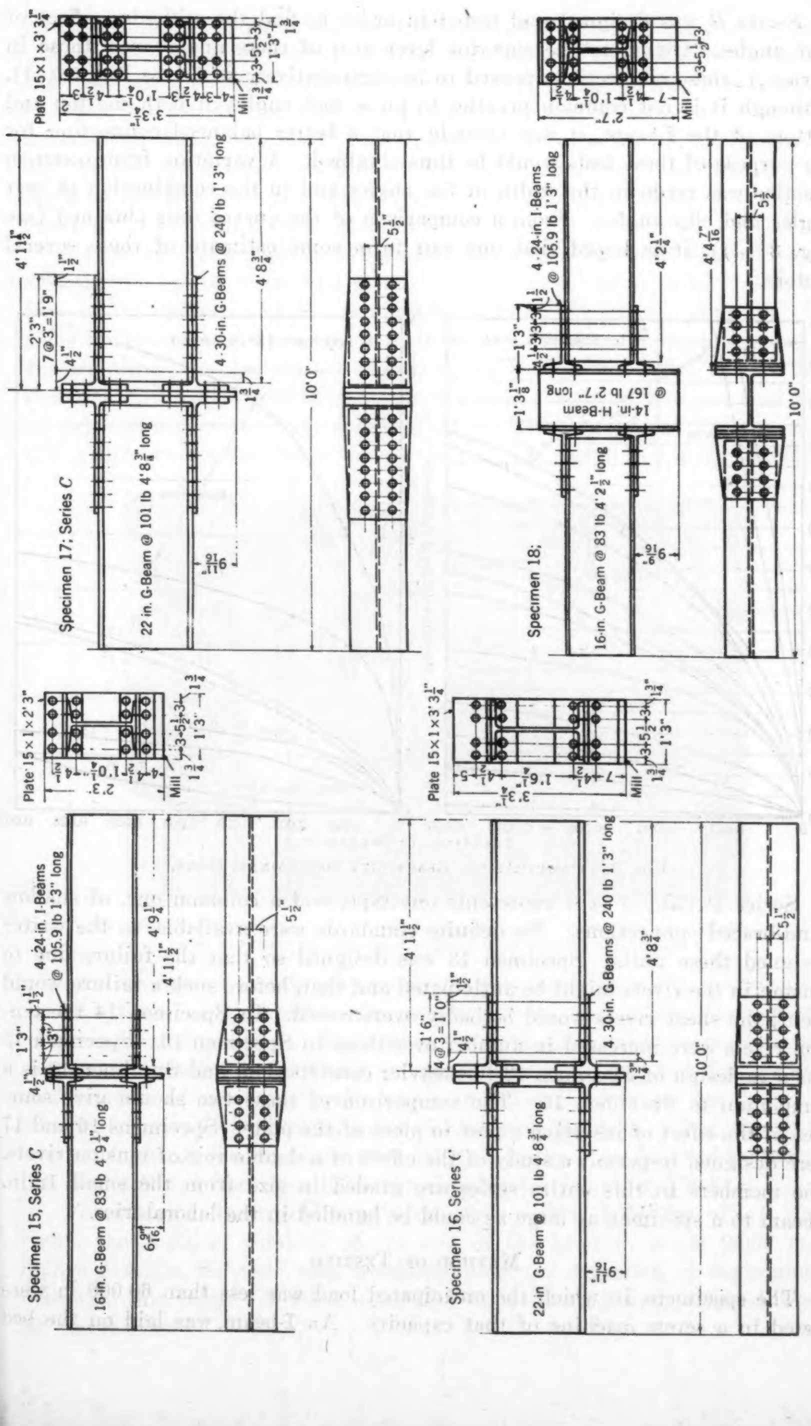


FIG. 2.—TEST SPECIMENS OF RIVETED CONNECTIONS, WITH 1-IN. φ-RIVETS, SERIES C.

Series *B* was designed and tested in order to find the stiffening effect of seat angles. Owing to the superior lever arm of these angles over those in Series *A*, these connections proved to be comparatively stiff (see Fig. 3 (*a*)). Although it is not common practice to place seat angles at both the top and bottom of the I-beam, it was thought that a better balanced connection for the purpose of these tests would be thus obtained. A variation from common practice was made in the width of the angles and in the combination of seat angles and clip angles. From a comparison of the curves thus obtained (see Fig. 3 (*a*)), it is hoped that one can form some estimate of these several factors.

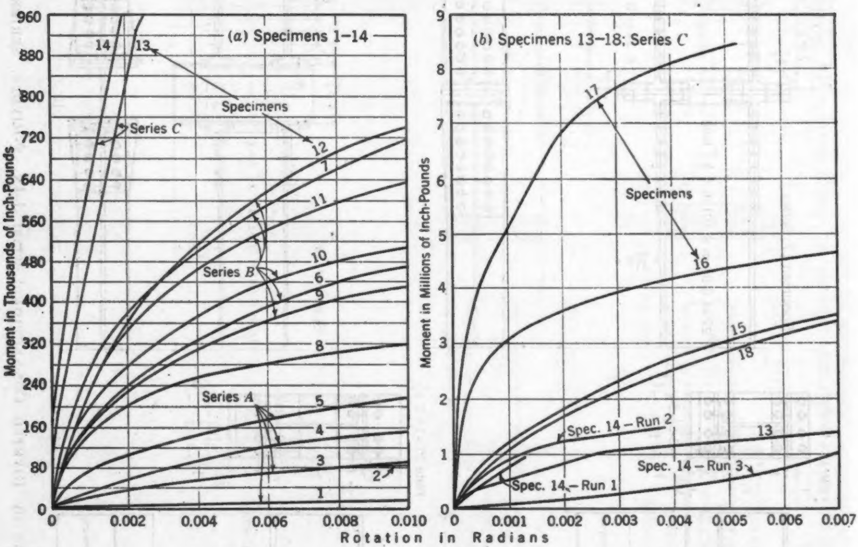


FIG. 3.—COMPARATIVE ELASTICITY OF CONNECTIONS.

Series *C* (Fig. 3 (*b*)) represents one type, and a common one, of shallow wind-braced connections. No definite standards were available, so the writer designed these units. Specimen 13 was designed so that the failure due to tension in the rivets might be anticipated and that, before such a failure would occur, the shear rivets would be badly overstressed. In Specimen 14 the tension rivets were increased in number over those in Specimen 13. Specimen 15 was a re-design of Specimen 13 for heavier construction, and Specimen 18 is a companion to Specimen 15. The comparison of these two should give some idea of the effect of inserting a post in place of the plate. Specimens 16 and 17 were designed to permit a study of the effect of a double row of tension rivets. The members in this entire series are graded in size from the small 12-in. I-beam to a specimen as large as could be handled in the laboratories.

METHOD OF TESTING

The specimens in which the anticipated load was less than 60 000 lb were tested in a screw machine of that capacity. An I-beam was laid on the bed

of the machine in order to support the bearing-blocks; the load was applied at the rate of 0.05 in, per min; and the machine was kept in balance, but the load was not increased, while the readings were being taken. For specimens in which the anticipated load was greater than the capacity of the 60 000-lb machine the tests were made in a hydraulic machine of 300 000-lb capacity, and the bed was extended as before. The supporting beam was from the same stock as the specimen and deflection readings on it permitted a comparison between the beam and the connection.

All the specimens were whitewashed so as to accentuate strain lines and deformation in the rivets and plates. During each test, a careful record was kept of the deflection of the center plate relative to the end supports. This aided the operator of the machine materially in judging the increments of load and the time at which the test should be stopped. The amount of this deflection was obtained by means of two Ames dials supported by a bar that rested on rods fastened to the ends of the specimen. The stem of the dial was in contact with a bracket secured to the central plate, as shown in Fig. 4. During the

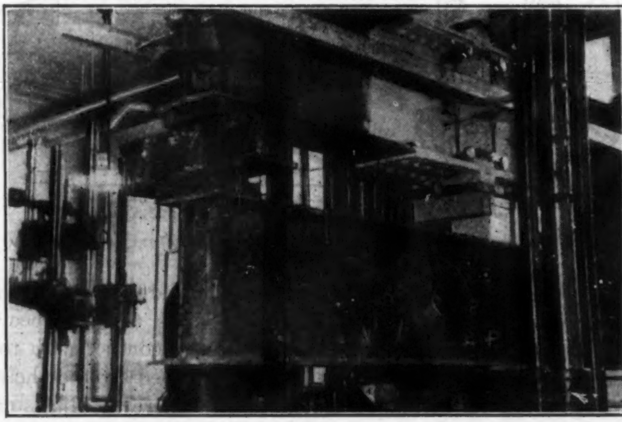


FIG. 4.—SPECIMEN 16, SHOWING POSITION OF THE SEVERAL DIALS AND METHOD OF BRACING A SPECIMEN AGAINST BUCKLING.

test the average of the two dial readings was plotted, but the resulting curve has not been incorporated in this paper; nor has it been used in computing the accompanying curves. For such a curve to be of value, corrections would have to be made for the shear deformation of the joint and for the elastic deformation of the beam itself. Although these corrections are appreciable, the dials proved advantageous for the purpose intended. For the tests of Series *A* and *B* the side bars were brass, $\frac{1}{4}$ in. by 1 in. in cross-section; and for Series *C* and Specimen 18, a 3-in. duralumen channel was used.

For a simple beam under uniform load, with a deflection of $\frac{1}{360}$ of its span, the angle of rotation of the end of the beam is about 0.009 radian. When possible, the tests were continued until the deflection of the connection exceeded this value. In a few cases this was prevented by the limited capacity of the machine.

Four dials, reading to 0.0001 in., were used in obtaining the angular rotation of each connection. In the case of Series *A* they were clamped directly to the flange of the I-beam with their stems resting against the central plate. This type of set-up could not be used in the remaining tests. In the case of Series *B*, the seat angles were in the way of the clamps, and, as some shear would occur between these angles and the I-beam, a different design was required. The flanges of the stubs of the I-beam in Series *C* interfered with the stem of the dial, and, in addition, the web of the stub interfered with the clamps as in Series *B*. The special holder shown in Fig. 5 was made to carry the dials in Series *B* and *C*. It was designed to furnish a three-point contact when clamped between the flanges of the I-beam. As first designed the pointed screws of the holder were made solid, but this proved unsatisfactory because the beam deformed under stress, causing one point to become loose. To correct this difficulty the points were placed on springs, as shown in Fig. 5(*b*).

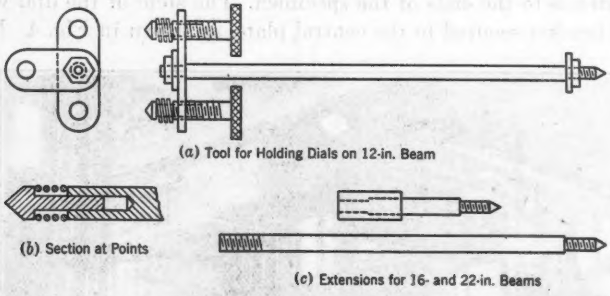


FIG. 5.—SPECIAL DEVICE FOR HOLDING DIALS THAT MEASURED ROTATION OF CONNECTION

In the tests of Series *B* the stems of the dials rested directly on the control plate. In the tests involvng the stub I-beam (Series *C*, and Specimen 18), the stems of the dials could not be brought in direct contact with the central plate, so steel bands of iron (1 in. by $\frac{1}{8}$ in.), were placed around the stubs and secured to the plate by set screws at the elevations where the dial stems would come in contact with them (see Figs. 4 and 6(*c*)). Care was taken that the bands did not touch any part of the specimen except through the set screws, thus assuring the condition that they were stationary relative to the plate.

In Series *B* and *C*, another set of dials was secured to the flange of the I-beam, but back of the connection, as shown in Figs. 4, 6(*b*), and 6(*c*). Readings from these dials were for the purpose of securing data that would be of value if an accident occurred to any of the other rotation dials during the test. The data thus obtained have not been incorporated in this paper. Fig. 6 shows schematically the location of the various dials and the general set-up of the tests.

DESCRIPTION OF TESTS AND FAILURES

Each specimen was tested by loading it on the central plate, thus inducing a moment of one-fourth the load times the span and a shear of one-half the load in each connection. Fig. 7(*a*) illustrates the manner in which connec-

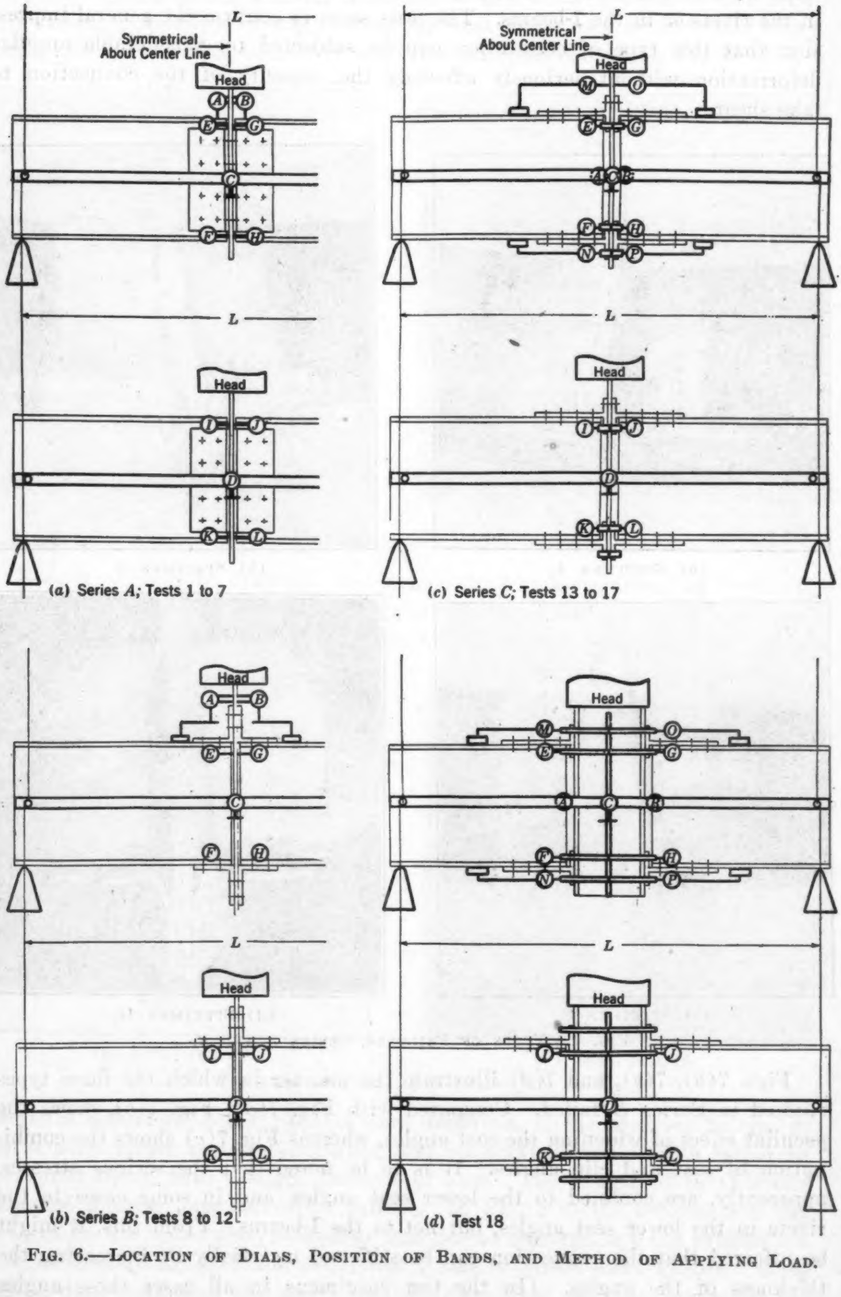
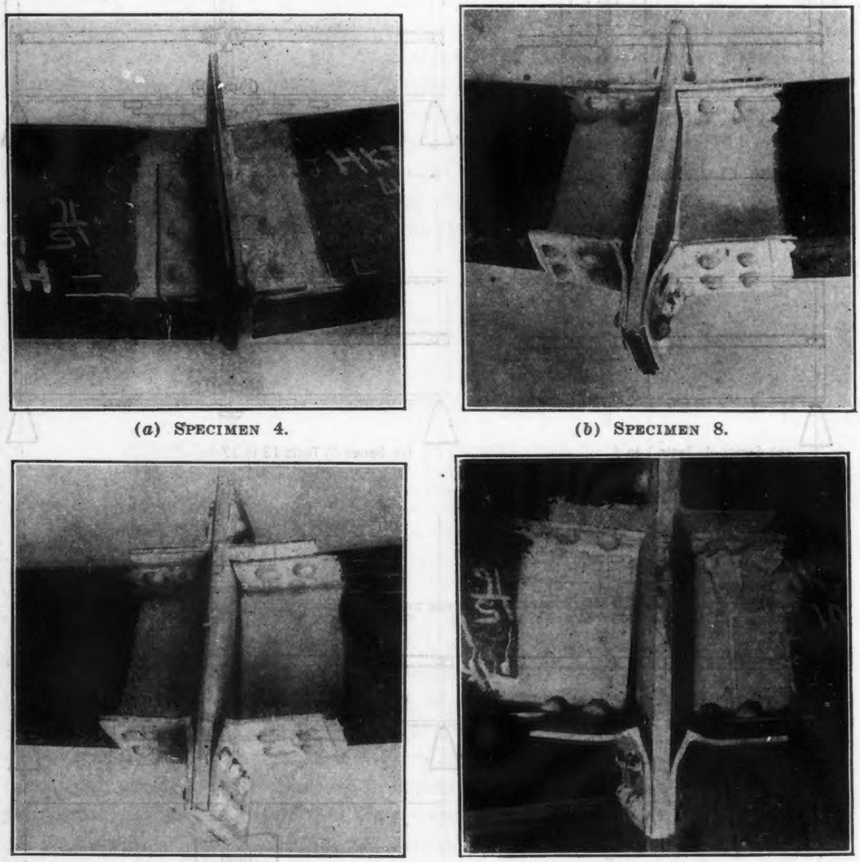


FIG. 6.—LOCATION OF DIALS, POSITION OF BANDS, AND METHOD OF APPLYING LOAD.

tions in Series *A* failed. The angles deformed badly without visible distress in the rivets or in the I-beams. The tests seem to confirm the general impression that this type of connection can be subjected to considerable angular deformation without seriously affecting the capacity of the connection to take shear.



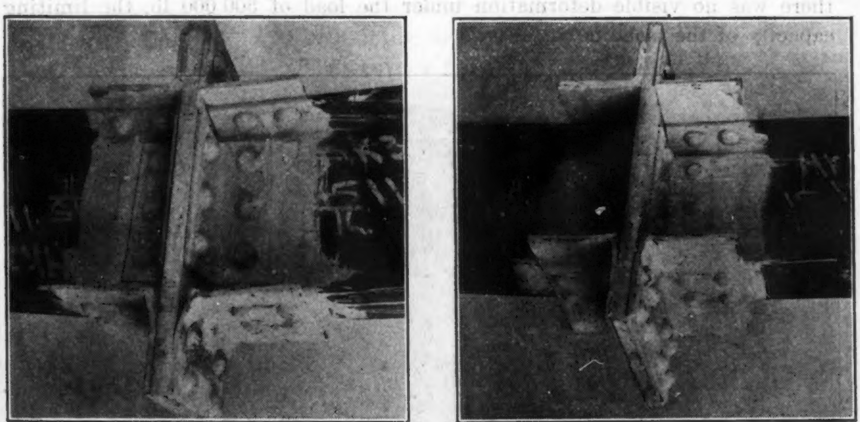
(*a*) Specimen 4. (*b*) Specimen 8.

(*c*) Specimen 9. (*d*) Specimen 10.

Fig. 7.—Types of Failures, Series *A* and *B*.

Figs. 7(*b*), 7(*c*), and 7(*d*) illustrate the manner in which the three types studied in Series *B* failed. Compared with Fig. 7(*d*), Fig. 7(*b*) shows the peculiar effect of widening the seat angles, whereas Fig. 7(*c*) shows the combination of seat and clip angles. It is to be noted that the serious stresses, apparently, are confined to the lower seat angles, and in some cases to the rivets in the lower seat angles, but not to the I-beams. From this, it might be inferred that the connection can be stiffened materially by increasing the thickness of the angles. (In the test specimens in all cases these angles

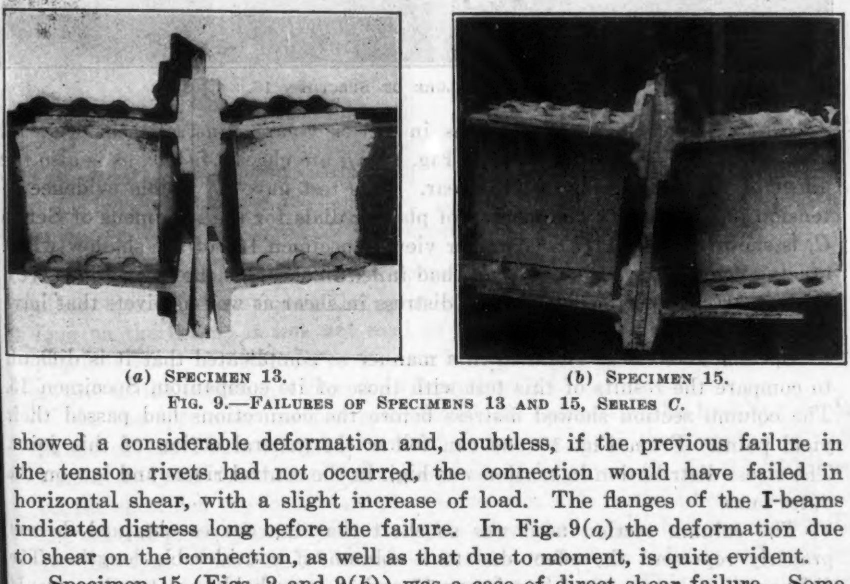
are $\frac{3}{8}$ in.) However, for a given deformation, this would lead to an increased unit stress. The effect of lengthening the seat angle may be noted by comparing the failure of Specimen 11 with that of Specimen 12, in Fig. 8.



(*a*) SPECIMEN 11. (*b*) SPECIMEN 12.

FIG. 8.—FAILURE OF SPECIMENS 11 AND 12, SERIES *B*.

Figs. 4, 9, and 10, illustrate the manner in which specimens of Series *C* failed. The rivets in Specimen 13 (Fig. 9(*a*)) failed in tension, and this connection showed weakness in the rivets early in the test. All shear rivets



(*a*) SPECIMEN 13. (*b*) SPECIMEN 15.

FIG. 9.—FAILURES OF SPECIMENS 13 AND 15, SERIES *C*.

showed a considerable deformation and, doubtless, if the previous failure in the tension rivets had not occurred, the connection would have failed in horizontal shear, with a slight increase of load. The flanges of the I-beams indicated distress long before the failure. In Fig. 9(*a*) the deformation due to shear on the connection, as well as that due to moment, is quite evident.

Specimen 15 (Figs. 2 and 9(*b*)) was a case of direct shear failure. Some tensile deformation of the rivets was clearly visible and the flanges of the

stub I-beam on the tension side were deformed. This connection failed suddenly. Specimen 16 (Fig. 2) failed by tension in the rivets, with some small deformation of the flanges of the I-beam, whereas in Specimen 17 (Fig. 2), there was no visible deformation under the load of 300 000 lb, the limiting capacity of the machine.

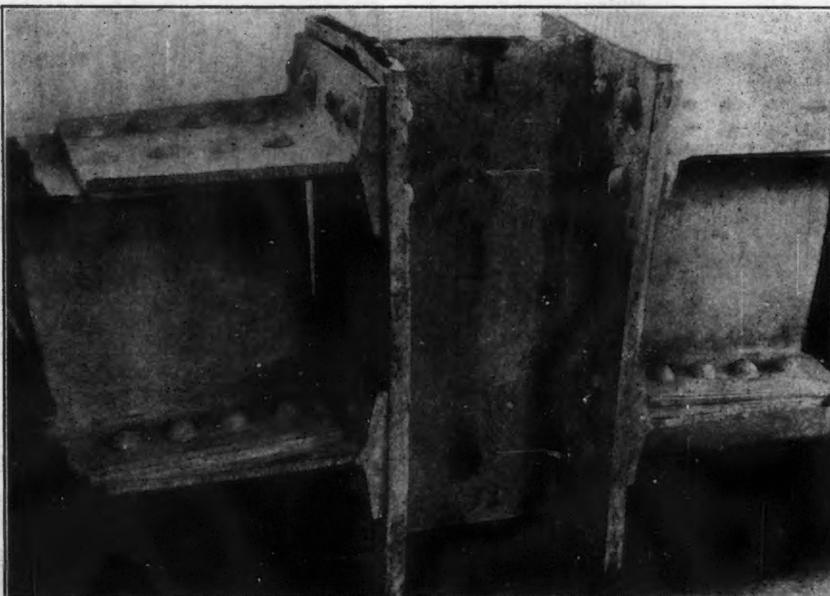


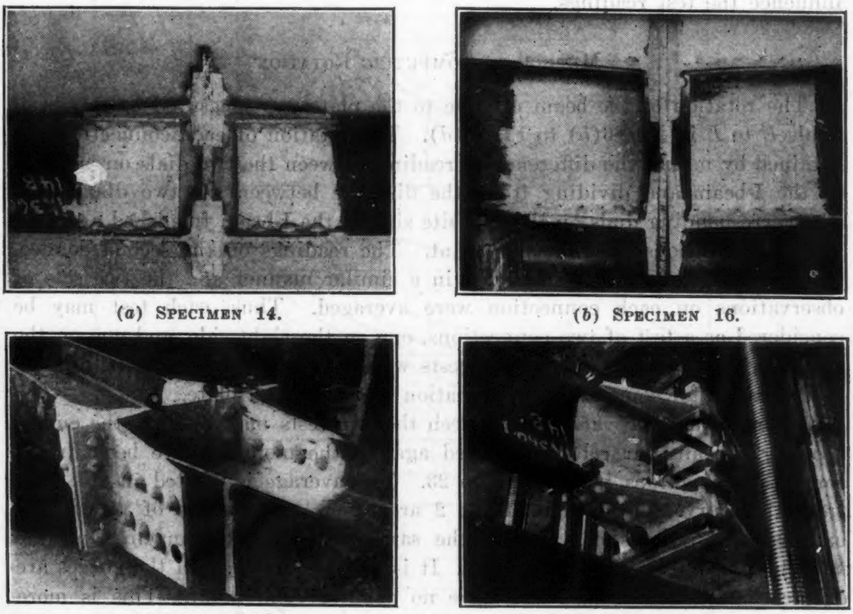
FIG. 10.—FAILURE OF SPECIMEN 18.

Sundry other failures of beams in Series *C* are illustrated in Fig. 11. The lines of shear in the I-beam of Fig. 11(*a*), are clearly visible, as is also the affect of the rivet heads due to shear. This test gave no visible evidence of tension in the rivets. The method of placing dials for the specimens of Series *C,* is shown in Fig. 11(*b*). Another view (Specimen 15) of the shallow wind-bracing connection, after the rivets had failed in shear, is shown in Fig. 11(*c*). In Fig. 11(*d*), Specimen 16 reveals distress in shear as well as rivets that have failed in tension.

Specimen 18 (Fig. 2) failed in a manner so complicated that it is difficult to compare the results of this test with those of its companion, Specimen 15. The column section showed distress before the connections had passed their yield point. From Fig. 10 one can follow the general action of this joint. The stress distribution in tension was high in the central rivets and low on the outer ones.

The column section used was so short that the curves obtained do not properly represent the effect due to a column of considerable length. The difference in the curves of Specimens 15 and 18 is probably greater than would be produced by the usual column section found in practice.

In Specimen 14 there was little distress evident due to rivet tension. Deformation due to shear on the rivets was visible and strain lines on the I-beam indicated that the connections had passed their limit of working stress. In testing this specimen, experience with Specimen 13 led the writer to anticipate a shear failure. After obtaining the stress reading for a load of 80 000 lb, the load was removed. This gave a set of curves that compare with the remainder of the series. The load was then transferred from the central plate to a point over the ends of the I-beam, thus relieving the shear on the stub beams, and the test was re-run to a load of 105 000 lb. In plotting the curve



(a) SPECIMEN 14.   (b) SPECIMEN 16.

(c) SPECIMEN 15.   (d) POSITION OF DIALS, SPECIMEN 14.

FIG. 11.—FAILURES OF SPECIMENS 14, 15, AND 16, SERIES *C*.

of the second run the effect of a decrease in the length of the lever arm was taken into consideration. The point of zero deflection was observed with no load on the beam; it was not read as the deflection due to the set from the previous load. Failure did not occur in the connections of this specimen but at one of the supports, where the I-beam buckled.

As the specimen had not been destroyed, except for the buckling of one flange of the I-beam, it was inverted and replaced in the machine in order that a study of the reversal of stresses might be obtained. In this third run on the specimen the loads were placed over the ends of the I-beam and not on the central plate, primarily because this edge of the plate had not been milled for this type of loading. The curve obtained was of an entirely different type from that of the others in this entire series of tests. This curve suggests a subject for further investigation.

In the case of the test of Specimens 15, 16, and 17, difficulty was encountered because of the tendency of the specimen to buckle (see Fig. 4). Stiffening angles were welded at the ends of the supporting beam, and similar angles were clamped to the ends of the specimen. In addition, angles were bolted on either side of the specimen. These angles were braced to the building to prevent a failure due to the upper ends of one of these sets of angles swinging to the north, whereas the upper ends of the angles on the opposite end of the test specimen swung to the south. Care was taken to ensure that the bracing did not interfere with the rotation of the test specimen on its supports, and thus influence the test readings.

Method of Computing Rotation

The rotation of the beam relative to the plate was measured by means of Dials E to L in Fig. 6(b) to Fig. 6(d). The rotation of each connection was obtained by noting the difference in reading between the two dials on one side of the I-beam and dividing it by the distance between the two dial stems. The corresponding dials on the opposite side of the I-beam furnished a second set of observations on this same point. The readings on the second connection of the specimen were obtained in a similar manner and the two sets of observations on each connection were averaged. Thus, each test may be considered as a test of two connections, one on the right side, and one on the left side, of the plate. These two tests were practically independent of each other, except for some slight deformation of the central plate.

In order that the variation between the two tests may be seen, the curves showing the angular rotation plotted against the moment have been drawn for both connections in Figs. 12 to 23. The average is plotted in full and, from this curve, the data in Table 2 are taken. The angle of rotation is expressed in radians or, what is the same thing, as its tangent, and the moment is expressed in inch-pounds. It is to be noted that all the curves are of the same general shape and have no definite yield point. This is more clearly shown in Series A and B than in Series C.

TABLE 2.—Constants from Curves in Figs. 11 to 18

Series	Test No. (1)	Slope tangent at origin, in $(10)^8$ inch-pounds (2)	Slope reloading curves, in $(10)^8$ inch-pounds (3)	Ultimate moment, in inch-pounds (4)	Series	Test No. (1)	Slope tangent at origin, in $(10)^8$ inch-pounds (2)	Slope reloading curves, in $(10)^8$ inch-pounds (3)	Ultimate moment, in inch-pounds (4)
A....	1	0.001	0.001	23 500	*B*....	10	4.7	1.33	1 055 000
	2	0.29	0.095	135 000		11	4.2	2.09	....
	3	0.39	0.126	159 500		12	4.4	1.75	1 026 000*
	4	0.28	....	403 000	*C*....	13	13.7	8.7	1 845 000
	5	0.47	....	529 000		14	8.8	8.8	1 640 000*
	6	2.9	....	1 225 000		15	21.3	20.0	5 700 000
	7	4.5	1.59	1 300 000		16	160.0	66.0	8 500 000*
B....	8	1.9	1.09	580 000		17	180.0	149.0	8 530 000*
	9	2.2	1.48	665 000					

* Test failed to reach the ultimate.

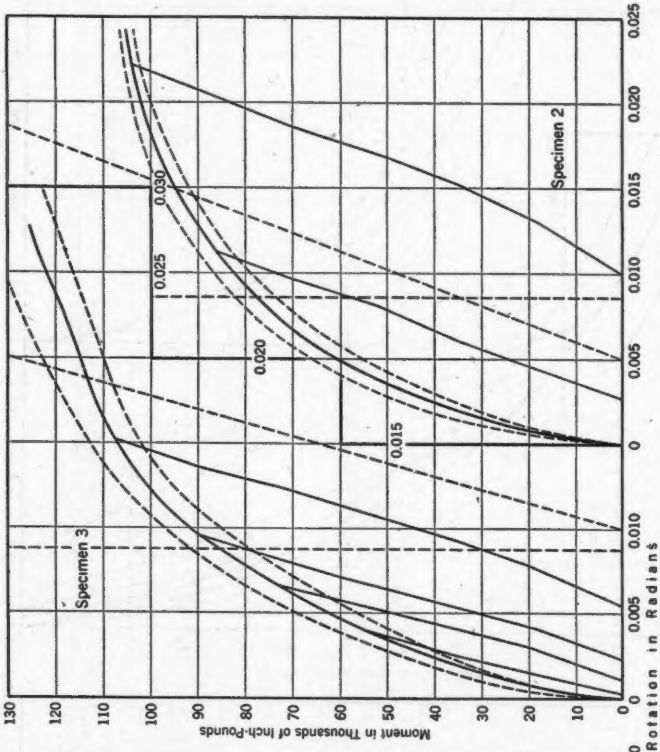


FIG. 13.—MODULUS OF ROTATION FOR SPECIMENS 2 AND 3, SERIES *A*.

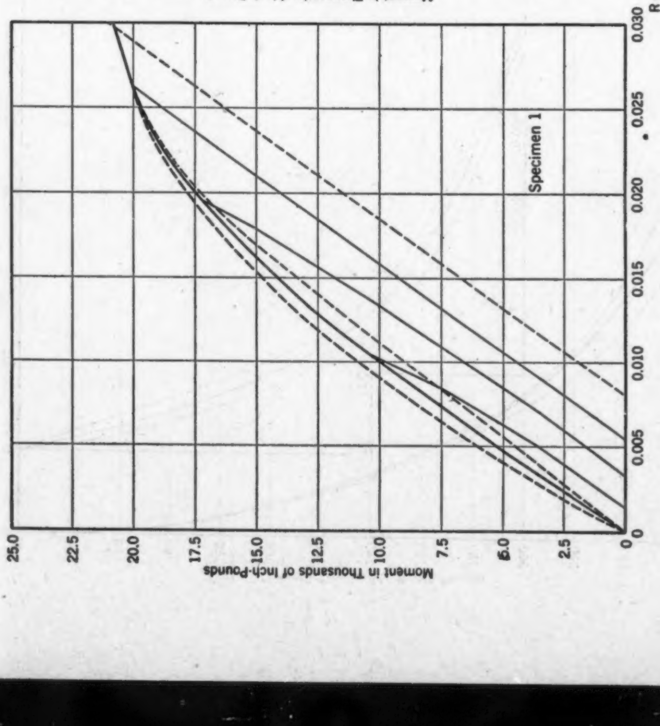


FIG. 12.—MODULUS OF ROTATION FOR SPECIMEN 1, SERIES *A*.

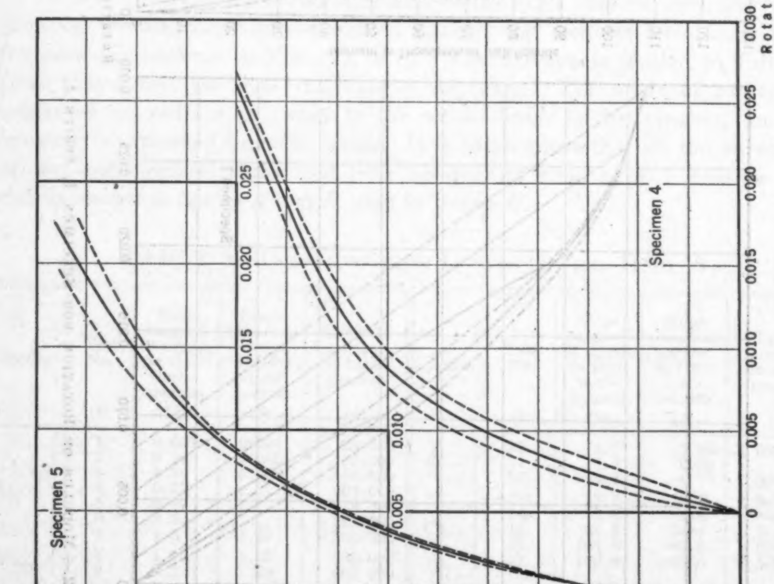


FIG. 14.—MODULUS OF ROTATION FOR SPECIMENS 4 AND 5, SERIES *A*.

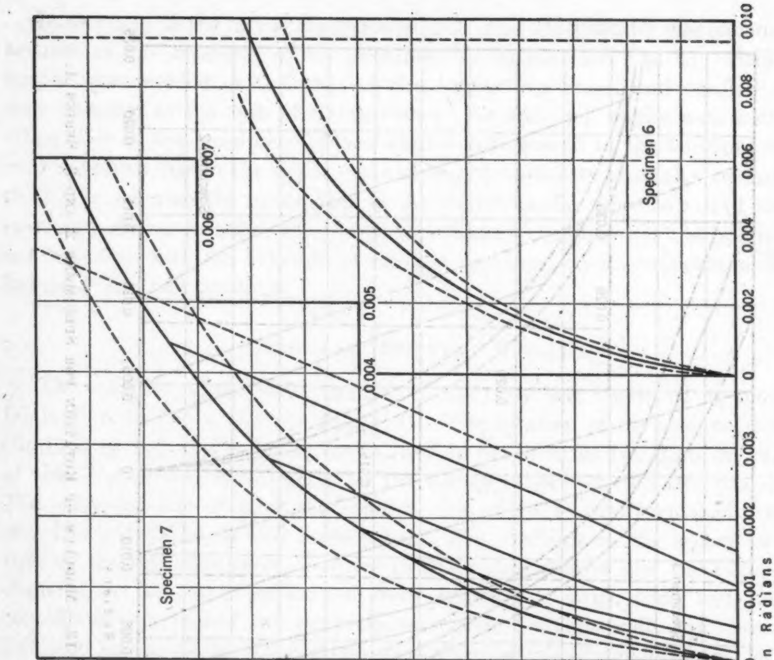


FIG. 15.—MODULUS OF ROTATION FOR SPECIMENS 6 AND 7, SERIES *A*.

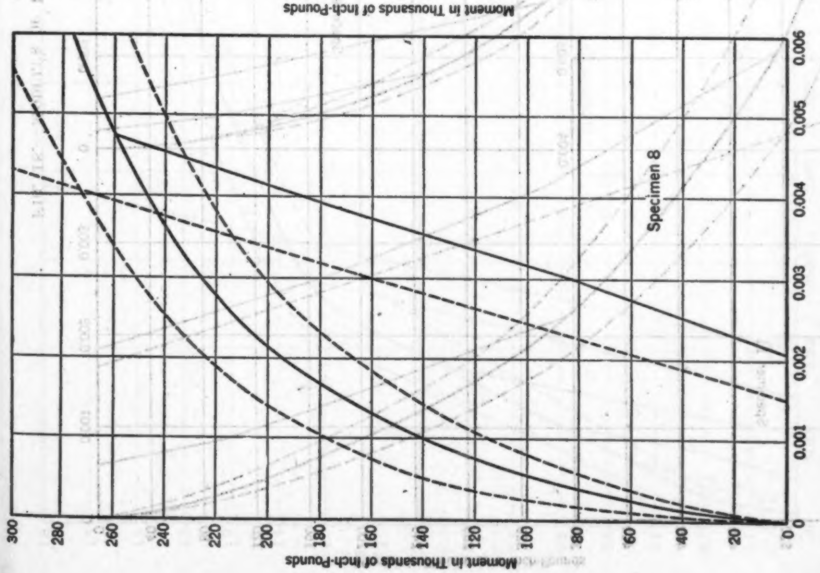


FIG. 16.—MODULUS OF ROTATION FOR SPECIMEN 8, SERIES *B*.

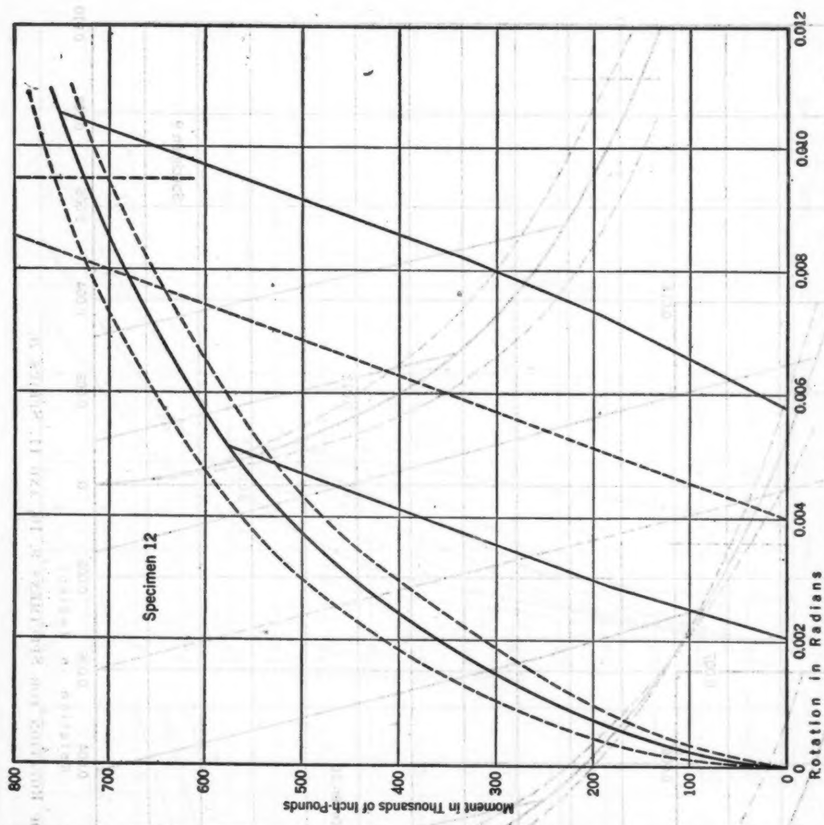


FIG. 17.—MODULUS OF ROTATION FOR SPECIMEN 12, SERIES *B*.

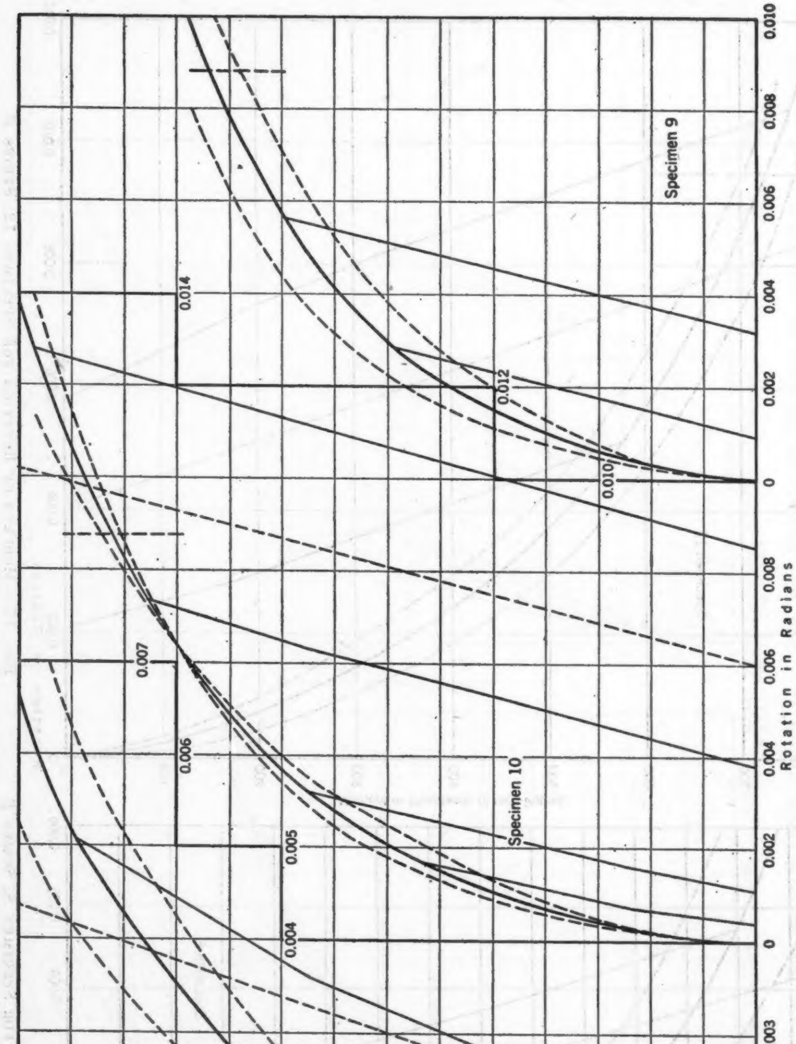


FIG. 18.—MODULUS OF ROTATION FOR SPECIMENS 9, 10, AND 11, SERIES *B*.

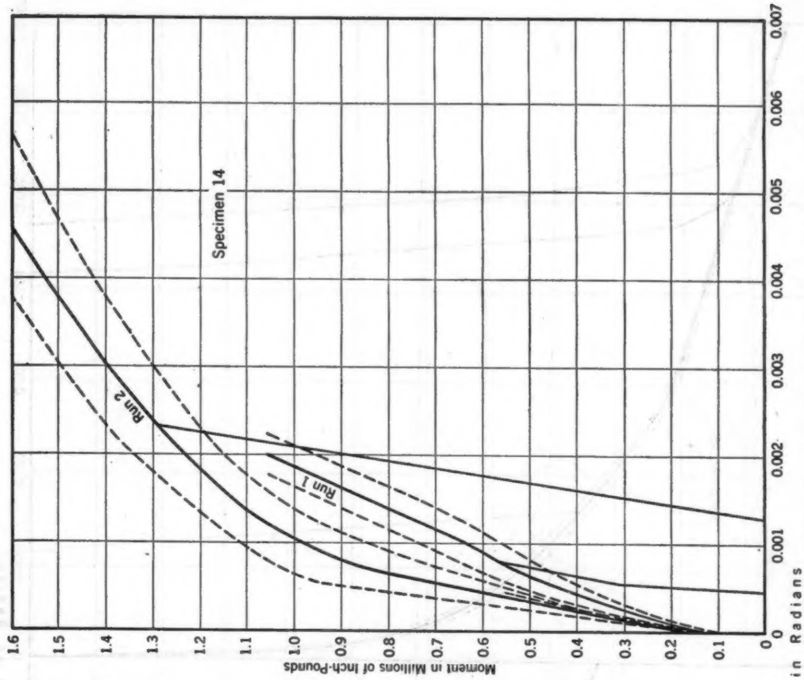


FIG. 20.—MODULUS OF ROTATION FOR SPECIMEN 14, SERIES *C*.

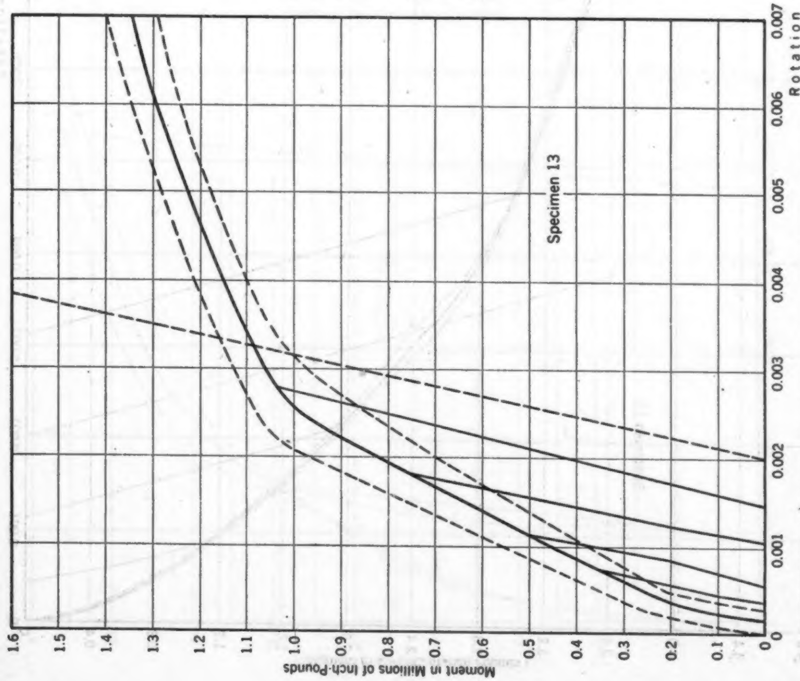


FIG. 19.—MODULUS OF ROTATION FOR SPECIMEN 13, SERIES *C*.

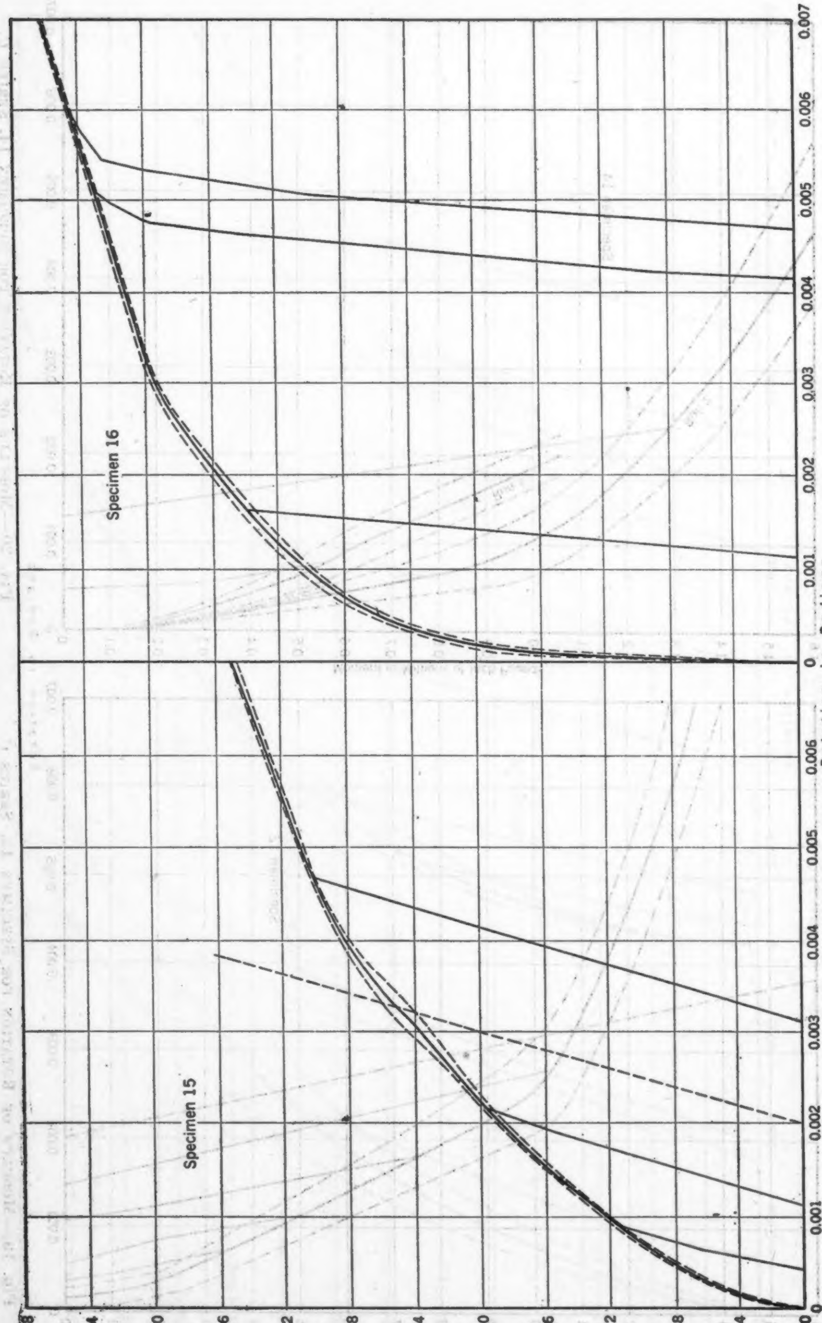


FIG. 21.—MODULUS OF ROTATION FOR SPECIMENS 15 AND 16, SERIES *C*.

FIG. 21.—MODULUS OF ROTATION FOR SPECIMENS 15 AND 16, SERIES *C*.

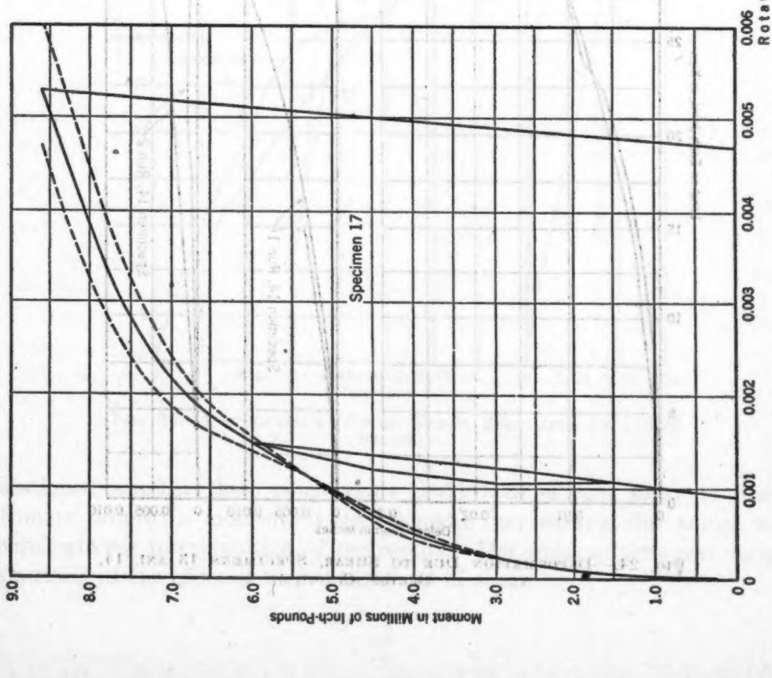


FIG. 22.—MODULUS OF ROTATION FOR SPECIMEN 17, SERIES *C*.

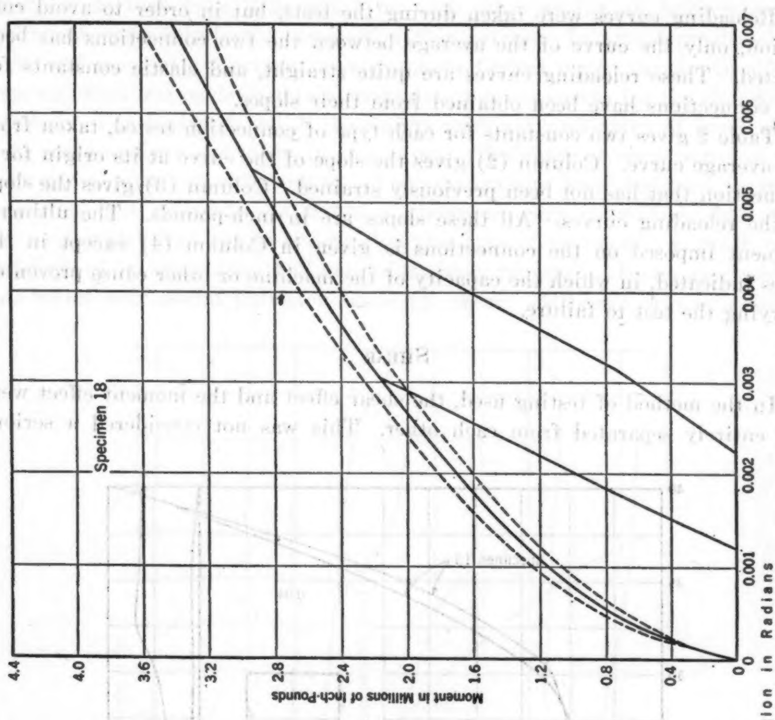


FIG. 23.—MODULUS OF ROTATION FOR SPECIMEN 18, SERIES *C*.

Reloading curves were taken during the tests, but in order to avoid confusion, only the curve of the average between the two connections has been plotted. These reloading curves are quite straight, and elastic constants for the connections have been obtained from their slopes.

Table 2 gives two constants for each type of connection tested, taken from the average curve. Column (2) gives the slope of the curve at its origin for a connection that has not been previously strained. Column (3) gives the slope of the reloading curves. All these slopes are in inch-pounds. The ultimate moment imposed on the connections is given in Column (4) except in the cases indicated, in which the capacity of the machine or other cause prevented carrying the test to failure.

SHEAR

In the method of testing used, the shear effect and the moment effect were not entirely separated from each other. This was not considered a serious

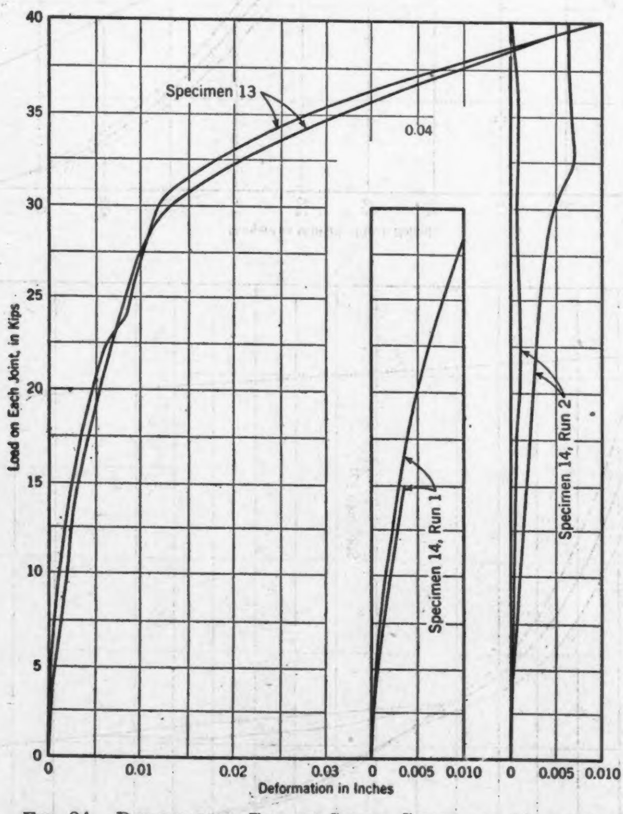


FIG. 24.—DEFORMATION DUE TO SHEAR, SPECIMENS 13 ANL 14, SERIES *C*.

objection as they have little effect on each other as far as deformation is concerned, especially in the range of working stress. In actual practice both shear and moment are present in a connection when under stress. In all the tests an effort was made to obtain stress-deformation curves for the connection in shear. For Series *A* and *B* the results were not considered of much practical value and are not given in this paper. The shear stresses were low and the results were not satisfactory. The connections were designed to resist shear, but not moment. In Series *C*, however, very interesting data were obtained, which lead one to believe that this type of connection is much more satisfactory in shear than has heretofore been supposed. In no specimen of this series was special provision made for shear. Although it was seriously

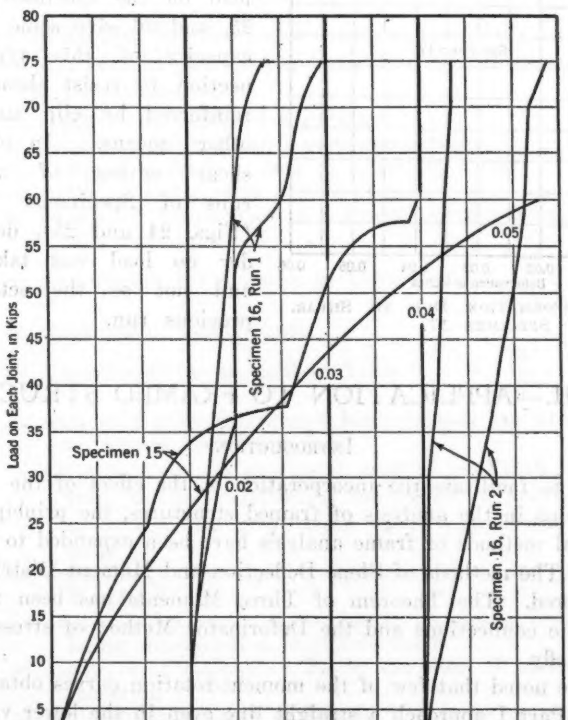


FIG. 25.—DEFORMATION DUE TO SHEAR, SPECIMENS 15 AND 16, SERIES *C*.

questioned whether these connections could take enough shear to produce ultimate failure in moment, it was thought that adding clip angles would complicate the interpretation of the results. The series of tests was designed as a study of the effect of moment and not of shear.

In Series C the shear dials were clamped to the I-beam as nearly midway between the flanges as possible, while the stem rested on a bracket secured to the central plate. All shear dials registered to 0.0001 in. The readings of these dials had to be corrected for the angular change of the I-beam due to rotation of the connection before the shear curves were plotted. The curves were drawn with the vertical displacement of the connection plotted against the load on the connection. This latter is one-half the load on the specimen. Figs. 24, 25, and 26 give some idea of the capacity of this type of connection to resist shear when not reinforced by clip angles, or by other means. In plotting the shear curves of the second runs of Specimens 14 and 16 (Figs. 24 and 25), deflection under no load was taken as zero and not as the set from the previous run.

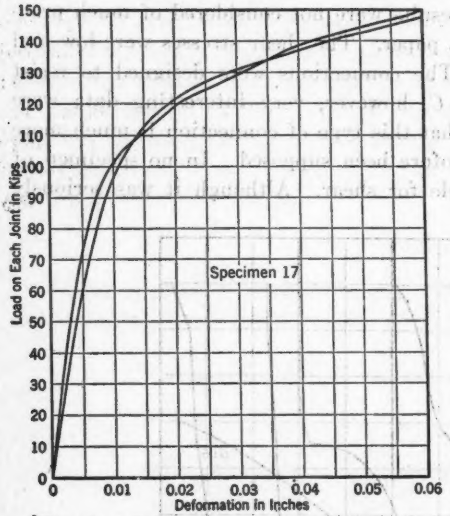


FIG. 26.—DEFORMATION DUE TO SHEAR, SPECIMEN 17.

PART II.—APPLICATION TO FRAMED STRUCTURES

INTRODUCTION

In order to facilitate the incorporation of the effect of the elasticity of the connections in the analysis of framed structures, the principal formulas of the several methods of frame analysis have been expanded to include this as a factor. The methods of Slope Deflection and Moment Distribution have been considered. The Theorem of Three Moments has been rewritten to include elastic connections and the Deformator Method of stress analysis is discussed briefly.

It is to be noted that few of the moment-rotation curves obtained experimentally in Part I approach a straight line even in the lower values. This makes it necessary to assume an arbitrary value for the moment on the connection, or for its angular rotation when substituting in the formulas involving the elasticity of connections. After the moment on this connection is computed it can be compared with the original assumption and the error corrected. Stated in another way, as the curves of Part I have no mathematical equations there seems to be no way of avoiding the trial-and-error method of computing when other than approximate results are desired.

For the purpose of yielding a first approximation and also to provide a comparison of the elastic properties of the several connections, the case of

a beam uniformly loaded and connected at each end to a rigid column for each of the connections listed, has been analyzed.

In expanding the formulas of the several methods it is assumed that the reader is familiar with these methods as applied to frames with rigid connections. The formulas found in texts on the subject are first given for comparison with the expanded form. Their derivation and use are omitted as they can readily be found in numerous books on the subject. These equations are followed by the corresponding formulas, expanded to include the effect of elasticity in the connections. The derivation of the expanded formulas is then given.

NOTATION

The notation adopted for this paper conforms essentially with the American Standard Symbols for Mechanics, Structural Engineering, and Testing Materials.[5] The definitions follow:

d = depth; total depth of a member.

h = heights of supports in a continuous beam.

i = width of member at base of groove in model.

j = width of rectangular groove in model.

l = length of the member of a model, corresponding to length, L, of a prototype.

q = a coefficient for moment in Table 3 $= -\dfrac{M}{w\,L^2}$.

t = thickness of model sheet.

u = a coefficient for moment in Table 3 $= -\dfrac{M}{w\,L^2}$.

w = unit uniformly distributed load, or load per unit length on a uniformly loaded member.

x = variable distances measured parallel to X-axis; $\bar{x}$ = distance from the left support to the center of gravity of Area A; $\bar{x}_1$ = distance from right support to center of gravity of Area A.

y = deflection; also, variable distances measured parallel to the Y-axis.

A = area of the moment curve due to transverse loads, the member being considered as a simple beam.

E = modulus of elasticity; E_c = modulus of elasticity of model material.

I = rectangular moment of inertia of a cross-section of a member.

K = stiffness factor.

L = length; nominal length of a member (distance from center to center of connections); L_1 and L_2 = transformed lengths of a member as defined in Equations (18) and (19); L_a = nominal length of the member in Span a; and L_{BC} = the length of the member in Span $B\,C$, transformed by the connection at Point B.

M = moment; M_c = moment at end of a loaded beam, with rigid connections and with length, L.

S = sections modulus of a cross-section of a member.

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[5] A. S. A.—Z 10*a*—1932.

Z = a coefficient of M such that $MZ = \theta_M$ = angle of rotation of the connection due to moment, M, values of Z being taken from Figs. 12 to 23.

α = ratio of the deflection of one end with respect to the other end, to the length of the member (Fig. 27).

$\beta = \theta_A - \alpha$ = slope at left end of a beam.

$\gamma = \theta_B - \alpha$ = slope at right end of a beam.

Δ = displacement; the displacement of the central support from a line joining the two end supports in a continuous beam.

θ = an angular distance; change in slope of the end tangent to the elastic curve; the angle of rotation of a connection; $\theta_M = MZ$ = angle of rotation of the connection due to moment, M, taken from Figs. 12 to 23.

σ = scale ratio of a model.

τ = ratio of characteristics of a model and its prototype; τ_r = ratio between two expressions for rigidity in model and prototype; τ_e = ratio between two expressions for elasticity in model and prototype.

Slope Deflection Method

The slope deflection method consists of the proper application of certain formulas to the analysis of rigid frames. When the connections are rigid the three fundamental formulas and their use are given in texts and treatises on the subject. The formulas for any beam, AB, are:

$$M_A = 2\,E\,\frac{I}{L}\,(2\,\theta_A + \theta_B - 3\,\alpha) - M_{cA} \quad \ldots\ldots\ldots\ldots(1)$$

and,

$$M_B = 2\,E\,\frac{I}{L}\,(2\,\theta_B + \theta_A - 3\,\alpha) + M_{cB} \quad \ldots\ldots\ldots\ldots(2)$$

and when the right end is hinged:

$$M_A = 3\,E\,\frac{I}{L}\,(\theta_A - \alpha) + \frac{3\,A\,\overline{x_1}}{L^2} \quad \ldots\ldots\ldots\ldots(3)$$

When the end connections are not rigid and allowance is made for the elastic properties of the connections, Equations (1) and (2) become:

$$M_A = 6\,E\,I\,\frac{2\,L_{2B}\,(\theta_A - \alpha) + L\,(\theta_B - \alpha)}{4\,L_{2A}\,L_{2B} - L^2} - M_{cA} \quad \ldots\ldots\ldots(4)$$

and,

$$M_B = 6\,E\,I\,\frac{2\,L_{2A}\,(\theta_B - \alpha) + L\,(\theta_A - \alpha)}{4\,L_{2A}\,L_{2B} - L^2} + M_{cB} \quad \ldots\ldots\ldots(5)$$

in which, $L_2 = L + 3\,E\,I\,Z$ and Z is a coefficient of M, such that $M\,Z = \theta_M$ = angle of rotation of the connection due to the moment, M. The signs of M_A, M_B, θ_A, θ_B, and α in Equations (1) to (5) are positive as indicated in Fig. 27; that is, the moment is positive when it tends to turn the member in a counter-clockwise direction. Angular changes are positive when they move in this same direction.

M_{cA} and M_{cB} are the end moments on the beam (considered fixed at the ends by rigid connections in Equations (1), (2), and (3), and by elastic

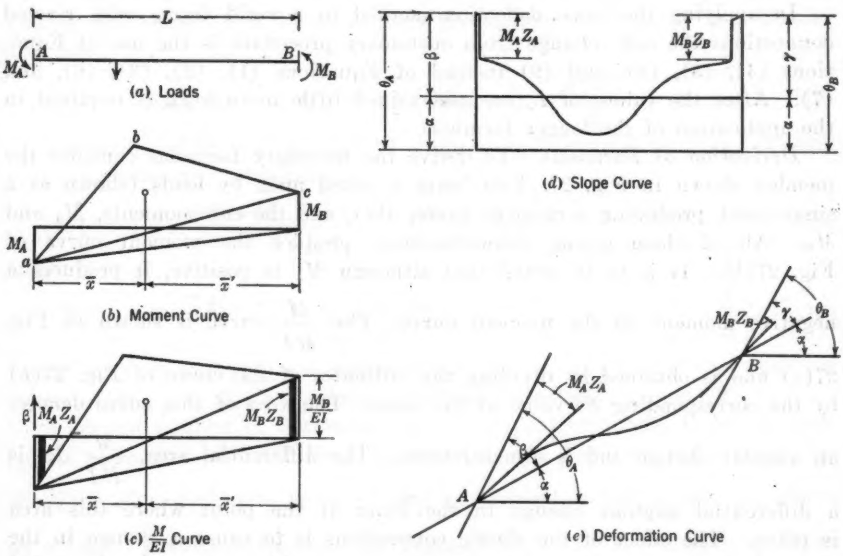

FIG. 27.

connections in Equations (4) and (5), due to external loads on the beam. For the rigid connections these values are:

$$M_{cA} = -\frac{2A}{L^2}(2L - 3\bar{x}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and,

$$M_{cB} = -\frac{2A}{L^2}(2L - 3\bar{x}_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and, for the beam with elastic connections, they are:

$$M_{cA} = -\frac{6A}{L} \times \frac{2L_{2B}\bar{x}_1 - L\bar{x}}{4L_{2A}L_{2B} - L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

and,

$$M_{cB} = -\frac{6A}{L} \times \frac{2L_{2A}\bar{x} - L\bar{x}_1}{4L_{2B}L_{2A} - L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

As M_{cA} and M_{cB} are the moments at the ends of a beam, their sign follows the convention of the beam theory (positive moment produces compression on the upper side of the beam) and not the convention designated by Fig. 27. By following the signs given in Equations (1) to (5), however, no confusion should arise from this source.

By replacing L_2 by L in Equations (4), (5), (8), and (9), Equations (1), (2), (6), and (7) result.

By making L_{2B} equal to infinity (a hinged connection at B) and L_{2A} equal to zero (a rigid connection at A) in Equations (4) and (8), Equation (3) results.

In applying the slope deflection method to a rigid frame with riveted connections the only change from customary procedure is the use of Equations (4), (5), (8), and (9) instead of Equations (1), (2), (3), (6), and (7). After the values of L_2 are ascertained little more work is required in the application of the longer formulas.

Derivation of Formulas.—To derive the necessary formulas consider the member shown in Fig. 27. This beam is acted upon by loads (shown as a single load, producing a moment curve, *abc*), and the end moments, M_A and M_B. All of them acting simultaneously, produce the moment curve of Fig. 27(*b*). It is to be noted that although M_A is positive, it produces a negative moment on the moment curve. The $\frac{M}{EI}$-curve is shown in Fig. 27(*c*) and is obtained by dividing the ordinates of the curve of Fig. 27(*b*) by the corresponding EI-value of the beam. The area of this curve denotes an angular change and is dimensionless. The differential area, $\frac{M}{EI}\,dx$, is a differential angular change in the beam at the point where this area is taken. The effect of the elastic connections is to cause a change in the slope of the beam at the point where the connection is located. As stated previously, this may be taken as $M\,Z$ and, like the area of the curve, is a dimensionless expression. It is represented on the curve, Fig. 27(*c*), by the heavy lines labeled $M_A\,Z$ and $M_B\,Z$.

From Fig. 27(*e*) it can be seen that the deflection of Point B from a tangent to the curve at Point A is $(\theta_A - \alpha)\,L$. It must be remembered that such angles as θ and α are very small and not as shown in the diagram, hence the tangent of the angle may be taken as equal to the angle. This same deflection can also be computed by taking the first moment of the $\frac{M}{EI}$-curve about Point B and adding thereto the angular change at the connection, A, multiplied by the length, L. Computing the deflection of Point B from the tangent at Point A by both methods and equating them:

$$-(\theta_A - \alpha)\,L = -\frac{M_A\,L}{2\,E\,I} \times \frac{2\,L}{3} + \frac{M_B\,L}{2\,E\,I} \times \frac{L}{3} - M_A\,Z_A\,L + \frac{A\,\overline{x_1}}{E\,I}$$

which may be simplified into:

$$(\theta_A - \alpha)\,L - \frac{2\,M_A\,L^2}{6\,EI} - M_A\,Z_A\,L + M_B\,\frac{L^2}{6\,EI} + \frac{A\,\overline{x_1}}{EI} = 0$$

Substituting the value of $L_2 = L + 3\,E\,I\,Z_A$:

$$(\theta_A - \alpha)\,L - \frac{2\,M_A\,L\,L_{2A}}{6\,EI} + \frac{M_B\,L^2}{6\,EI} + \frac{A\,\overline{x_1}}{EI} = 0 \quad \ldots\ldots(10)$$

Multiplying by $6\,E\,I$ and dividing by L, Equation (10) becomes:

$$2\,L_{2A}\,M_A - L\,M_B = 6\,E\,I\,(\theta_A - \alpha) + \frac{6\,A\,\bar{x}_1}{L} = 0 \;\ldots\ldots(11)$$

Taking the deflection of Point A from the tangent at Point B in a similar manner:

$$2\,L_{2B}\,M_B - L\,M_A = 6\,E\,I\,(\theta_B - \alpha) - \frac{6\,A\,\bar{x}}{L} = 0 \;\ldots\ldots\ldots(12)$$

The values of M_A and M_B can be obtained by solving Equations (11) and (12) simultaneously, obtaining:

$$M_A = 6\,E\,I\,\frac{2\,L_{2B}\,(\theta_A - \alpha) + L\,(\theta_B - \alpha)}{4\,L_{2A}\,L_{2B} - L^2} + \frac{6\,A}{L} \times \frac{2\,L_{2B}\,\bar{x}_1 - L\,\bar{x}}{4\,L_{2A}\,L_{2B} - L^2}\;.(13)$$

and,

$$M_B = 6\,E\,I\,\frac{2\,L_{2A}\,(\theta_B - \alpha) + L\,(\theta_A - \alpha)}{4\,L_{2B}\,L_{2A} - L^2} - \frac{6\,A}{L} \times \frac{2\,L_{2A}\,\bar{x} - L\,\bar{x}_1}{4\,L_{2B}\,L_{2A} - L^2}\;..(14)$$

Equations (13) and (14) are identical with Equations (4) and (5) when the values of M_{cA} and M_{cB} are substituted from Equations (8) and (9).

A second method of obtaining Equations (11) and (12) is by the conjugate beam method. The slope at the ends of a beam are the same as the reactions of a conjugate beam of the same length carrying the $\frac{M}{E\,I}$-curve as a load. Let these reactions be denoted by β and γ. Computing these values by taking moments first about Point B and then about Point A in Fig. 27(*c*) and noting from Fig. 27(*d*) that $\beta = \theta_A - \alpha$, and $\gamma = \theta_B - \alpha$, Equations (11) and (12) are obtained at once.

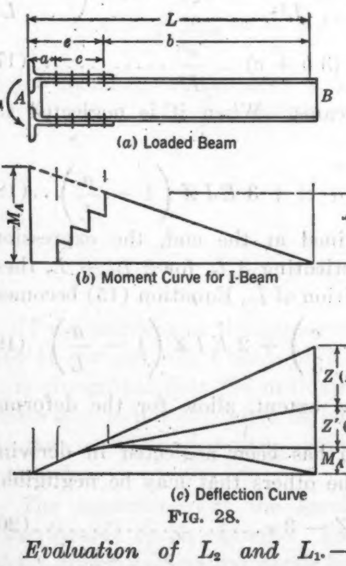

FIG. 28.

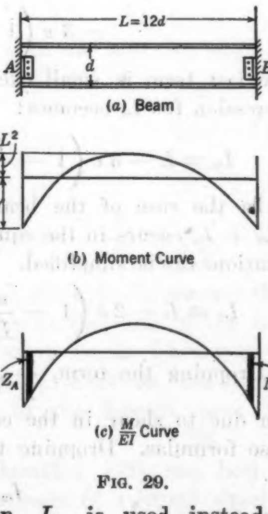

FIG. 29.

Evaluation of L_2 and L_1.—The expression, L_2, is used instead of $L + 3\,E\,I\,Z$ for the purpose of simplifying the formulas. As $3\,E\,I\,Z$ is

in units of length it may be visualized as the change in the length of the member due to the elasticity of the connections and L_2 may be visualized as the transformed length of the member.

In order to obtain a more accurate expression for L_2, that takes into account the stiffening effect of the connections of the types shown in Fig. 28(*a*), assume that the member is acted upon by a moment, M_A, at Point *A*. The moment at End *B* may be considered as zero. The moment on the I-beam is zero at the end where the instruments measuring *Z* were attached. The customary assumption is made that the shear (horizontal) is distributed equally between rivets and that the moment curve is straight in the connections instead of stepped as shown in Fig. 28(*b*). The angular change due to this stepped moment in the I-beam is, then, $M_A\,Z_{1A} = M_A\,\frac{b}{L} \times \frac{c}{2\,E\,I}$, approximately, and the total deflection of Point *B* from a tangent to the curve at Point *A* due to a moment, M_A, may be written as:

$$M_A\,Z_A\,(L - a) + M_A\,Z_{1A}\left(b + \frac{c}{3}\right) + M_A\,\frac{b}{L} \times \frac{b^2}{3\,E\,I} \;\ldots\ldots \quad (15)$$

which was assumed as $\frac{2\,M_A\,L\,L_2}{6\,E\,I}$ in deriving Equation (10).

Placing these two expressions equal will give a corrected definition for L_2:

$$L\,L_2 = 3\,E\,I\,Z_A\,(L - a) + 3\,E\,I\left(b + \frac{c}{3}\right)Z_1 + \frac{b^3}{L} \;\ldots\ldots (16)$$

Dividing by *L* and substituting the value of Z_1:

$$L_2 = 3\,E\,I\,Z\left(1 - \frac{a}{L}\right) + \frac{3}{2}\left(b + \frac{c}{3}\right)\frac{b\,c}{L^2} + \frac{(L - e)^3}{L^2} = L + 3\,E\,I\,Z\left(1 - \frac{a}{L}\right)$$

$$- 3\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{2\,L^2}\,(3\,b + c) - \frac{e^3}{L^2} \;\ldots\ldots\ldots\ldots (17)$$

The last term is small except for short beams. When it is neglected the expression for L_2 becomes:

$$L_2 = L - 3\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{2\,L^2}\,(3\,b + c) + 3\,E\,I\,Z\left(1 - \frac{a}{L}\right) \ldots (18)$$

In the case of the beam partly restrained at the end, the expression, $2\,L_2 + L$, occurs in the equations. By substituting $3\,L_1$ for $2\,L_2 + L$, these equations can be simplified. With this definition of L_1, Equation (15) becomes:

$$L_1 = L - 2\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{L^2}\left(b + \frac{c}{3}\right) + 2\,E\,I\,Z\left(1 - \frac{a}{L}\right) \ldots (19)$$

Dropping the term, $\frac{a}{L}$, would, to some extent, allow for the deformation due to shear in the connection which has been neglected in deriving these formulas. Dropping this term and the others that may be negligible:

$$L_2 = L + 3\,E\,I\,Z - 3\,e \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

and,

$$L_1 = L + 2\,E\,I\,Z - 2\,e \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

The Beam Partly Restrained at the Ends.—Probably the case that occurs the greatest number of times in steel design is that of the beam partly restrained at the ends by the connections. It is customary in designing such a beam to consider the connections as hinged. If, however, the ends are partly restrained by these connections the moment curve is as shown in Fig. 29(*b*), and a material saving can often be made. To find the value of the negative moment at the ends of the beam, Equations (8) and (9) are used in conjunction with Equations (4) and (5).

The beam with fixed ends is only a special case of the beam of Fig. 32, in which, θ_A, O_B, and α are all zero, and therefore the first terms of Equations (4) and (5) disappear.

The Partly Restrained Beam Under Uniform Load.—Of especial interest is the beam fixed at the ends by an elastic connection and loaded uniformly throughout its length. To find the value of the negative moment, the following values are substituted in Equation (8) or in Equation (9):

$$A = \frac{1}{8} w L^2 \times \frac{2}{3} L = \frac{1}{12} w L^3; L_{2A} = L_{2B} = L_2; \overline{x} = \overline{x}_1 = \frac{L}{2}$$

from which,

$$M_{cA} = M_{cB} = -\frac{6 w L^3}{12} \times \frac{(2 L_{2B} - L) \frac{L}{2}}{4 L^2_2 - L^2} = \frac{-w L^3}{4 (2 L_2 + L)} = -\frac{w L^3}{12 L_1} \quad (22)$$

in which, the value of L_1 is given in Equation (19).

The moment at the center of the span is:

$$\frac{1}{8} w L^2 + M_{cA} = \frac{1}{8} w L^2 - \frac{w L^3}{12 L_1} = \frac{1}{8} w L^2 \left(1 - \frac{2 L}{3 L_1}\right) \quad (23)$$

If the moments at the ends and center of this span are assumed equal to $u w L^2$ and $q w L^2$, thus defining u and q:

$$u = \frac{-L}{12 L_1} \quad (24)$$

and,

$$q = \frac{1}{8}\left(1 - \frac{2 L}{3 L_1}\right) \quad (25)$$

If the moment at the center of the span is not numerically greater than that at the end, the **I**-beam at the connection governs the limit of the load. It is recognized that the evaluation of the unit stress in the **I**-beam at the connection is not as simple as herein implied because, doubtless, the stresses in the connection govern, rather than those in the **I**-beam, near the connection.

The importance of the several terms of Equation (19) can best be demonstrated by an example. For this purpose a beam of a length equal to twelve times its nominal depth has been chosen. The load is assumed to be of sufficient value to give a fiber stress of 18 000 lb per sq in. Table 3 gives

TABLE 3.—Increase of Carrying Capacity Due to Connections, on a Uniformly Loaded Beam

Speci-men No.	Properties of Beam and Connections: Size of beam (1)	Moment of inertia, I, in inches (2)	Section modulus, S, in inches (3)	Distance, a, in inches (4)	Distance, e, in inches (5)	Length of beam L, in feet (6)	Computations for Moments: $1-\frac{a}{L}$ (7)	$2e\left(1-\frac{e}{L}\right)$, in inches (8)	$\frac{bc}{2L^2}\left(b+\frac{c}{3}\right)$ in inches (9)	Assumed moment, M_e, at beam connection, in inch-pounds (10)
	(a) Stress at Center Governs									
1	6-in. I-Beam @ 12.5 lb...	21.8	7.3	2.25	4.75	6.0	0.969	9	2	5 100
2	8-in. I-Beam @ 18.4 lb...	56.9	14.2	2.25	4.75	8.0	0.976	9	2	58 000
3	8-in. I-Beam @ 18.4 lb...	56.9	14.2	2.25	4.75	8.0	0.976	9	2	58 000
4	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	2.25	12.0	0.984	4+	....	95 000
5	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	150 000
6	18-in. I-Beam @ 54.7 lb...	795.5	88.4	2.25	2.25	18.0	0.990	4+	....	348 000
7	18-in. I-Beam @ 54.7 lb...	795.5	88.4	2.25	4.75	18.0	0.990	9	2.5	515 000
8	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	248 000
9	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	310 000
10	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	360 000
11	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	435 000
12	12-in. I-Beam @ 31.5 lb...	215.8	36.0	2.25	4.75	12.0	0.984	9	2	480 000
15	16-in. G-Beam @ 83.0 lb...	1 161.6	141.4	4.5	13.5	16.0	0.977	25	8	245 000
16	22-in. G-Beam @ 101.0 lb...	2 557.2	233.7	4.5	16.5	22.0	0.983	31	11	398 000
18	16-in. G-Beam @ 83.0 lb...	1 161.6	144.1	4.5	21.0	16.0	0.977	37+	13+	235 000
	(b) Stress at Ends Governs									
13	12-in. I-Beam @ 31.5 lb...	215.8	36.0	4.5	10.5	12.0	0.969	19+	5	648 000*
14	12-in. I-Beam @ 31.5 lb...	215.8	36.0	4.5	10.5	12.0	0.969	19+	5	648 000*
17	22-in. G-Beam @ 101.0 lb...	2 557.2	233.7	4.5	25.5	22.0	0.983	46	17	4 206 600*

TABLE 3.—(*Continued*)

Speci-men No.	Computations for Moments (*Continued*): Angular change, $M_e Z$, in connection (taken from curve) (11)	$2EIZ$, in inches (12)	$2EIZ\left(1-\frac{a}{L}\right)$, in inches (13)	Transformed length of beam, L_1, in inches (14)	Moment at center of beam, M_d, in inch-pounds $=18\ 000\,S$ (15)	$M_e = \frac{2L}{3L_1 - 2L}$ in inches (as a check on consumption) (16)	Load Computations: $W=\frac{+8M_d}{L}$ in pounds, load allowed on restrained beam (17)	Load allowed on beam restrained by connections (18)	Percentage excess load allowed (19)
	(a) Stress at Center Governs								
1	0.0049	1 260	1 220	1 285	131 400	5 100	14 600	15 170	4
2	0.0045	265	258	347	255 600	57 700	21 300	26 100	22
3	0.0045	265	258	347	255 600	58 000	21 300	26 100	22
4	0.0045	613	603	743	648 000	95 500	36 000	41 300	15
5	0.0044	379	373	512	648 000	150 000	36 000	44 300	23
6	0.00435	592	586	796	1 591 200	349 000	59 000	71 860	22
7	0.00412	384	380	589	1 591 200	515 000	59 000	78 000	32
8	0.0041	224	211	348	648 000	248 000	36 000	49 700	38
9	0.00395	165	161	298	648 000	310 000	36 000	53 200	48
10	0.0037	133	131	268	648 000	360 000	36 000	56 000	55
11	0.00345	103	101	238	648 000	436 000	36 000	60 200	67
12	0.0034	91.5	90	227	648 000	475 000	36 000	62 400	73
15	0.00315	90	88	263	2 593 800	2 460 000	108 100	210 600	95
16	0.00315	121	119	363	4 206 600	3 980 000	127 600	248 100	94
18	0.0035	104	101	269	2 593 800	2 360 000	108 100	206 400	91
	(b) Stress at Ends Governs								
13	0.0015	30	29	159	425 000	....	36 000	59 600	66
14	0.00105	21	20	150	363 000	....	36 000	56 100	56
17	0.00055	20	20	255	1 890 000	....	127 600	184 700	45

* $M_e = 18\ 000\,S$, in inch-pounds.

the results of the computations for the several connections tested in Part I. In computing the first fifteen connections in Table 3 the values of M_u were assumed, the values of $M\ Z$ were read from the curve, after which L_1 was computed. From the formula, $M_u = M_q \left[\frac{-2L}{3L_1 - 2L}\right]$ (obtained from Equations (24) and (25)), the assumed values were checked. In these cases the values of M_q were taken as 18 000 S, in which, S is the section modulus of the beam. In the case of Specimens 13, 14, and 17, the stress at the end of the beam rather than that at the center was the determining factor. The moment, M_q, was computed from the reciprocal formula, $M_q = M_u\left(1 - \frac{3L_1}{2L}\right)$. From Fig. 29 it is seen that $-M_u + M_q = \frac{1}{8}\ w\ L^2$, the formula customarily used in solving this problem. The increase of strength obtained by considering the connection as elastic over the hinged connection is given by the expression, $\frac{-M_u + M_q}{18\,000\ S}$. This is given in Column (19), Table 3.

Theorem of Three Moments

The theorem of three moments may be written (see Fig. 30):

$$\frac{L_a}{I_a} M_A + 2\left(\frac{L_a}{I_a} + \frac{L_b}{I_b}\right) M_B + \frac{L_b}{I_b} M_C + \frac{6\ A\ \overline{x_a}}{L_a I_a} + \frac{6\ A\ \overline{x_b}}{L_b I_b}$$
$$= 6\ E\ \Delta\left(\frac{1}{L_a} + \frac{1}{L_b}\right) \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Equation (26) is found in many forms all of which can be transformed into the one here given or can be derived as special cases of it. If the

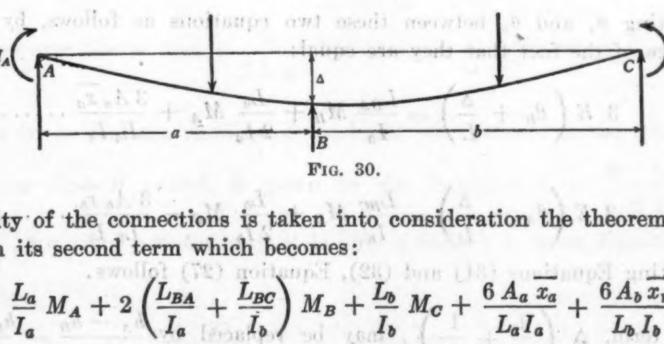

Fig. 30.

elasticity of the connections is taken into consideration the theorem changes only in its second term which becomes:

$$\frac{L_a}{I_a} M_A + 2\left(\frac{L_{BA}}{I_a} + \frac{L_{BC}}{I_b}\right) M_B + \frac{L_b}{I_b} M_C + \frac{6\ A_a\ \overline{x_a}}{L_a I_a} + \frac{6\ A_b\ \overline{x_{1b}}}{L_b I_b}$$
$$= 6\ E\ \Delta\left(\frac{1}{L_a} + \frac{1}{L_b}\right) \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

This theorem (Equation (26) and Equation (27)) may be considered as a special case readily derived from the slope deflection equations. The theorem ex-

presses only the relationship between the loads on a continuous beam on three supports, the elastic properties of the beam, and the moment at the deflection of the central support. The end moments are considered as loads.

To derive Equation (27), it is only necessary to note that the slope of Span *AB* at Point *B*, is the same as the slope of Span *BC* at the same point.

Span *AB* may be considered as hinged at the left end and elastically connected at the right, while Span *BC* is hinged at the right and elastically connected at the left end. For a beam with a hinge at the left end, Equation (14) becomes:

$$M_B = \frac{3\,E\,I\,(\theta_B - \alpha)}{I_{2B}} - \frac{3\,A\,\overline{x}}{L\,L_{2B}} \quad\ldots\ldots\ldots\ldots\ldots(28)$$

when L_2 becomes infinite, while for a beam with a hinge at the right end, Equation (13) becomes:

$$M_A = \frac{3\,E\,I\,(\theta_A - \alpha)}{L_{2A}} + \frac{3\,A\,\overline{x}_1}{L\,L_{2A}} \quad\ldots\ldots\ldots\ldots\ldots(29)$$

For Span *AB* substitute the following in Equation (28); Thus, $M_{AB} = M_{BC}$;

$A\,\overline{x} = A_a\,\overline{x}_a + \frac{M_A\,L^2_a}{6}$; and $\alpha = -\frac{\Delta}{L_a}$. Then,

$$L_{BA}\,M_B = 3\,E\,I_a\left(\theta_B + \frac{\Delta}{L_a}\right) - \frac{3\,A_a\,\overline{x}_a}{L_a} - \frac{M_A\,L_a}{2}$$

Similarly, substitute in Equation (27) for Span *BC*, $M_A = -M_{BA}$ $A\,\overline{x}_1 = A_b\,\overline{x}_b + \frac{M_C\,L^2_b}{6}$, and $\alpha = \frac{\Delta}{L_b}$, obtaining:

$$-\,L_{BC}\,M_B = 3\,E\,I_b\left(\theta_A - \frac{\Delta}{L_b}\right) + \frac{3\,A_b\,\overline{x}_b}{L_b} + \frac{M_C\,L_b}{2} \quad\ldots\ldots(30)$$

Eliminating θ_A and θ_B between these two equations as follows, by taking advantage of the fact that they are equal:

$$3\,E\left(\theta_B + \frac{\Delta}{L}\right) = \frac{L_{BA}}{I_a}\,M_B + \frac{L_a}{2\,I_a}\,M_A + \frac{3\,A_a\,\overline{x}_a}{L_a\,I_a} \quad\ldots\ldots\ldots(31)$$

and,

$$3\,E\left(\theta_A - \frac{\Delta}{L}\right) = \frac{L_{BC}}{I_b}\,M_B + \frac{L_b}{2\,I_b}\,M_C - \frac{3\,A_b\,\overline{x}_b}{L_b\,I_b} \quad\ldots\ldots\ldots(32)$$

Subtracting Equations (31) and (32), Equation (27) follows.

The term, $\Delta\left(\frac{1}{L_a} + \frac{1}{L_b}\right)$, may be replaced by $\frac{h_A - h_B}{L_a} - \frac{h_B - h_C}{L_b}$

This comes from the geometry of the figure, in which, h_A, h_B, and h_C are the heights of the several supports, the assumption being made that they are in line when the beam is not strained.

For the special case in which there is no settlement of supports and the beam carries a uniform load, Equation (27) becomes:

$$\frac{L_a}{I_a} M_A + 2\left(\frac{L_{BA}}{I_a} + \frac{L_{BC}}{I_b}\right) M_B + \frac{L_b}{I_b} M_C + \frac{w L^3_a}{4 I_a} + \frac{w L^3_b}{4 I_b} = 0 \ldots\ldots (33)$$

MOMENT DISTRIBUTION METHOD

In the moment distribution method of analyzing rigid frames, three groups of formulas are required: (1) The moment induced at either end of a beam by loads when the beam is considered as fixed-ended; (2) the moment induced at one end of a beam by a moment imposed at the other; and (3) the rigidity of the beam with respect to a moment at the end.

Equations (6) and (7), and the last term of Equation (3), fulfill the requirements for the first group when no correction is made for the yielding of the connections; Equations (8) and (9) fulfill the requirements when this factor is taken into consideration.

In the second group of formulas, the carry-over factor is 0.500 in the case of rigid connections. In the case of elastic connections this fact can be obtained from Equations (13) and (14). Let the beam of Fig. 27(a) be the one under consideration; let $\theta_B = 0$; $A = 0$; and $\alpha = 0$; then:

$$M_A = 6\,E\,I \,\frac{2\,L_{2B}\,\theta_A}{4\,L_{2A}\,L_{2B} - L_2} \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

and,

$$M_B = 6\,E\,I \,\frac{L_{2A}\,\theta_A}{4\,L_{2B}\,L_{2A} - L_2} \ldots\ldots\ldots\ \ldots\ldots\ldots (35)$$

Eliminating θ_A between Equations (34) and (35):

$$M_B = M_A \,\frac{L}{2\,L_{2B}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

The carry-over factor then is $\frac{L}{2\,L_{2B}}$ which reduces to 0.500 when L_{2B} is equal to L.

In order to find the rigidity of the beam with respect to the moment at End A or End B (which is given by the formulas, $\theta_A = \frac{M_A\,L}{4\,E\,I}$ and $\theta_B = \frac{M_B\,L}{4\,E\,I}$, for rigid connections), it is only necessary to solve Equation (34) for θ, thus,

$$\theta_A = M_A\,\frac{4\,L_{2A}\,L_{2B} - L^2}{12\,L_{2B}\,E\,I} = M_A\left(\frac{L_{2A}}{3\,E\,I} - \frac{L^2}{12\,L_{2B}\,E\,I}\right)$$

$$= \frac{M_A}{3\,E\,I}\left(L_{2A} - \frac{L^2}{4\,L_{2B}}\right) \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

and,

$$\theta_B = \frac{M_B}{3\,E\,I}\left(L_{2B} - \frac{L^2}{4\,L_{2A}}\right) \dots\dots\dots\dots\dots\dots (38)$$

Equations (37) and (38) are the formulas from which the stiffness of the beam may be determined when computing the unbalanced moment distribution about a joint. If $L_2 = L$ in Equations (37) and (38) the usual formulas for both ends rigidly connected obtain. If End A is hinged and End B is rigid, Equation (38) reduces to $\theta_B = \frac{M_B L}{3 E I}$, as $L_{2B} = L$ and $L_{2A} = $ infinity.

Examples.—The changes in the moment distribution method due to elastic connections can best be illustrated by comparing the solution of a problem in which all connections are rigid with that of one in which some of the connections are taken as elastic.

Consider the beam of Fig. 31 supporting a uniform load, w, of 100 lb per ft. The connections at Points A and D are hinged and the beam, at

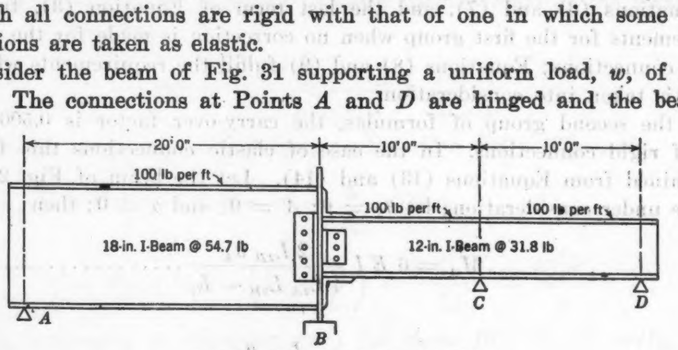

FIG. 31.—A CONTINUOUS BEAM.

Point C, is continuous. The beam will first be considered as continuous at Point B, and, later, the case of elastic connections at Point B will be solved. These two problems are analyzed in a parallel manner so that a comparison can readily be made between them.

Rigid Connections.—The fixed-end moments, M_o, for this case (Fig. 31) are:

$$M_{BA} = -\frac{3\,A\,\bar{x}}{L^2} = -\frac{3 \times 48 \times 10^5}{240} = -60 \times 10^3$$

$$M_{BC} = M_{CB} = -\frac{2\,A\,\bar{x}}{L^2} = -\frac{2 \times 6 \times 10^5}{120} = -10 \times 10^3$$

and,

$$M_{CD} = -\frac{3\,A\,\bar{x}}{L^2} = -\frac{3 \times 6 \times 10^5}{120} = -15 \times 10^3$$

The carry-over factor is 0.5000 for both ends of Beam BC. The stiffness factors for the several beams are evaluated as follows:

$$K_{BA}\,E\,\theta = \frac{3I}{L}\,E\,\theta = \frac{(3)\,(795.5)}{240}\,E\,\theta = 9.93\,E\,\theta$$

$$K_{CB}\,E\,\theta = M_{BC} = \frac{4\,I}{L}\,E\,\theta = \frac{(4)\,(215.8)}{120}\,E\,\theta = 7.2\,E\,\theta$$

and,

$$K_{CD}\,E\,\theta = \frac{3I}{L}\,E\,\theta = \frac{(3)\,(215.8)}{120}\,E\,\theta = 5.4\,E\,\theta$$

These values are used in the computations shown in Fig. 32.

Elastic Connections.—It is first necessary to compute L_2 for Beams *BA* and *BC*. In both cases the moments are low and the tangent at the origin gives $\dfrac{M}{MZ}$ with considerable accuracy. Therefore, the values given in Table 2 will be taken for $\dfrac{1}{Z}$.

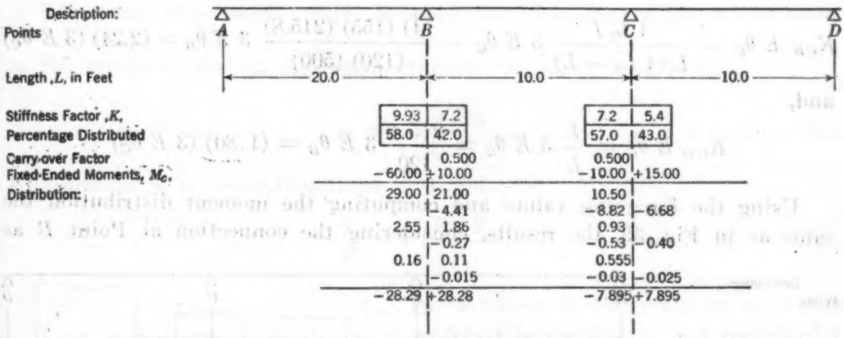

FIG. 32.—COMPUTATIONS FOR BEAM IN FIG. 31, CONSIDERING THE CONNECTION AT POINT *B* AS RIGID.

From Equation (18):

$$L_{2BA} = 240 - 7.5 + \frac{(3)\,(3)\,(10)^7\,(795.5)}{(10)^8\,(2.9)} = 232.5 + 246.5 = 479$$

$$L_{2BC} = 120 - 14.5 + 3.5 + \frac{(3)\,(3)\,(10)^7\,(215.8)}{(10)^8\,(4.2)} = 109 + 46 = 155$$

$$A\,\overline{x}_{AB} = \frac{w\,L^3}{24} = \frac{(100)\,(20)\,(240)^3}{24} = (1\,152)\;(10)^6$$

$$A\,\overline{x}_{BC} = A\,\overline{x}_{CB} = \frac{(100)\,(10)\,(120)^3}{24} = (72)\;(10)^6$$

and the fixed moment, M_c,

$$M_{BA} = \frac{-3\,A\,\overline{x}}{L\,L_{2BA}} = \frac{-(3)\,(1\,152)\,(10)^6}{(240)\,(479)} = -\;29\,900$$

$$M_{BC} = \frac{-6\,A\,\overline{x}}{L\,(4\,L_2 - L)} = \frac{-(6)\,(72)\,(10)^6}{(120)\,(620 - 120)} = -\;7\,200$$

$$M_{CB} = \frac{-6\,A\,\overline{x}\,(2\,L_2 - L)}{L^2\,(4\,L_2 - L)} = \frac{-(6)\,(72)\,(10)^6\,(120)}{(120)^2\,(500)} = -\,11\,400$$

$$M_{CD} = \frac{-3\,A\,\overline{x}}{L^2} = -\;15\,000$$

The carry-over factor is 0.50 for the right end of Beam BC, but for the left end, it is $\frac{L}{2\,L_2} = \frac{120}{(2)\,(155)} = 0.387$.

The stiffness factors, K, for the several beams are:

$$K_{BA}\,E\,\theta_B = \frac{I}{L_2}\,(3\,E\,\theta_B) = \frac{795.5}{479}\,(3\,E\,\theta_B) = (1.66)\,(3\,E\,\theta_B)$$

$$K_{BC}\,E\,\theta_B = \frac{4\,I}{4\,L_2 - L}\;3\,E\,\theta_B = \frac{4\,(215.8)}{620 - 120} = (1.73)\,(3\,E\,\theta_B)$$

$$K_{CB}\,E\,\theta_C = \frac{4\,L_2\,I}{L\,(4\,L_2 - L)}\;3\,E\,\theta_C = \frac{(4)\,(155)\,(215.8)}{(120)\,(500)}\;3\,E\,\theta_B = (2.24)\,(3\,E\,\theta_C)$$

and,

$$K_{CD}\,E\,\theta_C = \frac{I}{L}\;3\,E\,\theta_C = \frac{215.8}{120}\,3\,E\,\theta_B = (1.80)\,(3\,E\,\theta_C)$$

Using the foregoing values and computing the moment distribution the same as in Fig. 32, the results, considering the connection at Point B as

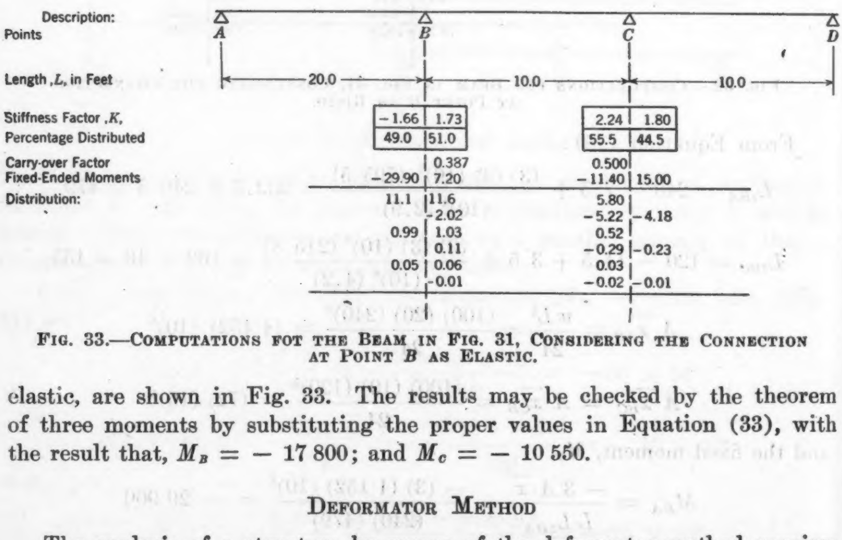

FIG. 33.—COMPUTATIONS FOT THE BEAM IN FIG. 31, CONSIDERING THE CONNECTION AT POINT B AS ELASTIC.

elastic, are shown in Fig. 33. The results may be checked by the theorem of three moments by substituting the proper values in Equation (33), with the result that, $M_B = -\,17\,800$; and $M_C = -\,10\,550$.

DEFORMATOR METHOD

The analysis of a structure by means of the deformator method requires a model cut from a sheet of celluloid or other elastic material. In designing this model, if deformation depends primarily on bending as it does in building frames, the moment of inertia of the various parts must be proportional to those of the structure. This proportion also applies to the connections. The widths of the various parts of the model, as d, are computed so that the cubes of these widths are proportional to the moments of inertia of the corresponding members. The rigidity of each member is proportional

to $\frac{EI}{L}$ whereas that of the corresponding member in the model is $\frac{E_c\,t\,d^3}{12\,l}$, in which, E_c is the modulus of elasticity of the model material; l is the length of the member; and t is the thickness of the sheet of celluloid. If the ratio between these two expressions for rigidity is τ_r, then, $\tau_r = \frac{E_c\,t\,d^3}{12\,l} \div \frac{E\,I}{L}$; or,

$$\tau_r = \frac{E_c\,t\,d^3\,L}{12\,l\,E\,I} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

The same ratio should exist between the elasticity of the connections in the structure and in the model, whence: $\tau_r = \frac{E_c\,t\,i^3}{12\;j} \div \frac{1}{Z}$; or,

$$\tau_r = \frac{E_c\,t\,i^3\,Z}{12\,j} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

Eliminating τ_r between Equations (39) and (40),

$$\frac{j}{i^3} = \sigma\,\frac{E\,I\,Z}{d^3} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

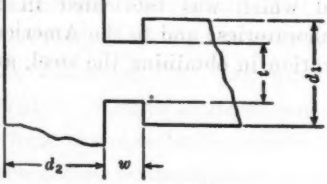

FIG. 34.

In which, σ is the scale ratio of the model; j is the width of a rectangular groove cut in the member at the location of the connection; and i is the width of the member at the base of this groove (see Fig. 34).

SUMMARY AND CONCLUSIONS

The tests described in Part I give the elastic properties of eighteen connections of three distinct types (see Figs. 12 and 26, and Table 2). The tests showed that the connections of Series *A* can be deformed through angles considerably beyond working conditions without affecting their capacity to resist shear. The connections of Series *B* possessed this property in a much smaller degree, but are capable of deforming within the working range. The deformations in Series *A* and *B* are due to the yielding of the angles rather than to the rivets or main members. The connections shown as Series *C* are comparatively rigid. Their method of failure can be controlled by detail of design.

Part II indicates the method of analyzing structural frames when consideration is given to the elastic properties of the connections. Expressions for the "transformed length of the member", L_s and the "transformed length of the beam", L_1, have been developed and incorporated into the formulas for stress analysis by the slope deflection, moment distribution, and deformator methods, as well as into the theorem of three moments. The

saving resulting from utilizing the resistance of the connections in designing a simple beam is indicated in Table 3, where it is shown that the load in some cases may be increased more than 70% by utilizing the support given by the connections.

Appreciation

Acknowledgment is freely given to the following for material assistance in the preparation of this paper: G. E. J. Pistor, M. Am. Soc. C. E., through whose interest the steel for these tests was made available and also for many helpful suggestions in planning the tests; Frederick Skene, Dean of the School of Technology, College of the City of New York, for use of their laboratories, equipment, etc., and for his co-operation; A. H. Beyer, O. E. Hovey, and Jonathan Jones, Members, Am. Soc. C. E., for many helpful suggestions during the gathering of the data and the writing of the paper; J. Sanford Peck, Assoc. M. Am. Soc. C. E., for material assistance, co-operation, and many suggestions during the conduct of the tests; A. G. Hayden, M. Am. Soc. C. E., and Mr. C. W. Vanderbilt of Kalman Steel Company for material assistance and helpful suggestions in the preliminary stages of the investigation; Mr. Leo S. Pavelle for his assistance as a photographer; Mr. Arthur Gatterdam for the suggestion of the use of L_a and for a careful review of the derivation of the equations; the McClintic-Marshall Company for furnishing the steel which was fabricated in its Bethlehem (Pa.) shops and delivered to the laboratories; and to the American Institute of Steel Construction for its co-operation in obtaining the steel, and for many suggestions.

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PAPERS

ANALYSIS OF THICK ARCH DAMS, INCLUDING ABUTMENT YIELD

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Synopsis

A graphical solution of stresses in a circular arch under the influence of normal loads is presented herein. These curves differ from those previously published, in that they include the effect of abutment yield. Furthermore, a finite value for Poisson's number is used rather than the infinite value previously assumed for simplication.

Although no novel method of analyzing arches is introduced, the derivation of the final stress equations is traced because it combines advances made by Cain, Jakobsen, Vogt, and others. The forms of the equations are greatly altered, in order to facilitate computations and the plotting of graphs.

Some examination is made to determine the quantitative effect of variations in the necessary assumptions of the values of the modulus of elasticity of rock, and Poisson's number. An example of the use of the curves is appended.

Introduction

In the past several years, great strides toward a more rational solution of the stresses in an arch dam have been made by numerous contributors. Of necessity, the authors of these papers assumed either fixed or hinged-end conditions, and precise mathematical relations were built up on this basis. In an effort to consolidate the gains made in this particular field of indeterminates, F. H. Fowler, M. Am. Soc. C. E., prepared a series of graphs,[2] from which the stresses at the critical sections of crown and abutment may be obtained. That such a graphical procedure is necessary for general practical use may be verified easily by examining the complex equations arising from the mathematical treatment.

Note.—Discussion on this paper will be closed in April, 1935, *Proceedings*.

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[2] "A Graphic Method of Determining Stresses in Circular Arches under Normal Loads by the Cain Formulas," by F. H. Fowler, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 92 (1928), p. 1513.

For relatively thin arches, the assumed end conditions do not introduce an appreciable error. However, it is generally recognized that, for relatively thick arches, the assumptions give stresses that can only be considered as approximate.

That the final reliance in the design of such a structure as an arch dam must rest upon the judgment of an experienced designer can scarcely be gainsaid. However, although an exact solution of the stresses in so statically an indeterminate structure is almost never to be hoped for, the judgment of the designer can be assisted more adequately by an analysis that includes all the known factors of the same magnitude of importance than by one that neglects one or two such factors.

Quantitatively, light was first shed on the bothersome subject of abutment yield by Fredrik Vogt, Assoc. M. Am. Soc. C. E., in 1924, and again in 1927.[3] Unfortunately, the stress equations arising from the inclusion of this factor and Poission's effect, are many times more unwieldy and impractical for general design use than the equations that exclude these factors. That the complexity of these equations should be mastered is evidenced by the fact that, in many instances, there may be a possible saving of concrete when the arch is analyzed on the basis of yielding abutments, because of the more equable distribution of stress between crown and abutment.

Discussion of the division of load between cantilevers and arches is not pertinent to this paper because it was possible to construct the graphs so that the curves would be applicable regardless of the proportion of water load determined, or assumed, to be carried by the arches. Wherever possible, detailed derivation is eliminated and reference is made to the origin.

Notation.—In the notation advanced herein (see Appendix), an effort has been made to reconcile the conflicting symbols introduced by several writers, with the American Standard Symbols for Mechanics, Structural Engineering, and Testing Materials.[4]

Derivation of Stress Equations

Arch Ring Deformations.—Using the customary sign conventions (that is, linear deformations are positive when they are elongations; stresses are positive when they produce a positive deformation; and, water pressure, p_e, is taken as positive), Cain and Jakobsen derive the relation for an arch ring, shown in Fig. 1,[5] as follows:

$$\Delta_C = X\left\{\frac{r^2_n}{E_c\, I_n}\left[\, r_C\,(\phi_1 - \sin\phi_1) - r_n\left(\sin\phi_1 - \frac{\phi_1}{2} - \frac{\sin 2\,\phi_1}{4}\right)\right]\right.$$
$$\left. + \frac{r_n}{E_c\, t}\left[\, 1.94\,\phi_1 - 0.47\sin 2\,\phi_1\right]\right\} - \frac{\sigma}{E_c}\, p_e\, r_e\, r_n \sin\phi_1 \ldots\ldots (1)$$

[3] "Ueber die Berechnung der Fundamentdeformation," by Fredrik Vogt, Assoc. M. Am. Soc. C. E., Math. Naturv. Klasse 1925, No. 2; also "Stresses in Thick Arches of Dams": Discussion by Professor Vogt, *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 554.

[4] A.S.A.—Z10a—1932.

[5] "Stresses in Thick Arches of Dams," by B. F. Jakobsen, *Transactions*, Am. Soc. C. E., Vol. 90 (1927), pp. 475 and 522.

and,

$$\psi_C = r_C\, \phi_1 - r_n \sin \phi_1 \dots\dots\dots\dots\dots\dots\dots (2)$$

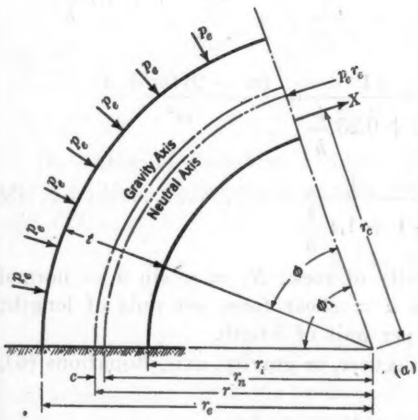

FIG. 1.

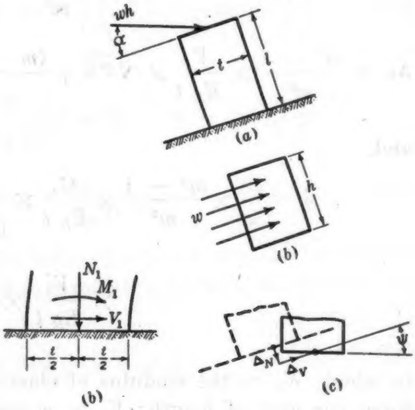

FIG. 2.

The following pertinent relations also are derived:

$$M = X\ (r_C - r_n \cos \phi) \dots\dots\dots\dots\dots\dots\dots (3)$$

$$N = p_e\, r_e - X \cos \phi \dots\dots\dots\dots\dots\dots\dots (4)$$

and,

$$V = X \sin \phi \dots\dots\dots\dots\dots\dots\dots\dots (5)$$

in which, M, N, and V are the moment, thrust, and shear, respectfully, at any arbitrary section at an angle, ϕ, from the crown.

Foundation Deformations.—Although the ideal case, upon which Professor Vogt's foundation deformation equations are based,[3] can never be exactly attained in practice, the quantitative information derived is probably correct within a usable range of accuracy. A gratifying check on his theoretical and empirical forms has been made by the Committee of Engineering Foundation on Arch Dam Investigation.[6]

In Fig. 2, Professor Vogt presents a hypothetical cantilever of length, l, and thickness, t. In plan, the loaded area is equal to h. Under the influence of the load, $w\,h$ (assuming plane sections to remain plane), any deformation of the foundation may be resolved into the components, Δ_{n1}, Δ_v, and ψ. In

[6] Rept. by the Committee of Engineering Foundation on Arch Dam Investigation, Vol. II, May, 1934, pp. 225 to 232, inclusive.

terms of N_1, V_1, and M_1, at the base, their values are as follows:

$$\Delta_N = \frac{m^2 - 1}{m^2} \times \frac{N_1}{E_R\, t} \times \sqrt[3]{t^2 h} \quad \text{.................}(6)$$

$$\Delta_V = \frac{m^2 - 1}{m^2} \times \frac{V_1}{E_R\, t} \times \sqrt[3]{t^2 h} + \frac{(m-2)(m+1)}{m^2} \times \frac{M_1}{E_R\, t} \times \frac{1}{1 + 1.1\frac{t}{h}} \quad ..(7)$$

and,

$$\psi = \frac{18}{\pi} \times \frac{m^2 - 1}{m^2} \times \frac{M_1}{E_R\, t} \times \frac{1}{1 + 0.25\frac{t}{h}} + \frac{(m-2)(m+1)}{m^2}$$

$$\times \frac{V_1}{E_R\, t} \times \frac{1}{1 + 1.1\frac{t}{h}} \quad \text{.....................}(8)$$

in which, E_R = the modulus of elasticity of rock; $N_1 = w \sin \alpha$ = normal force per unit of length; $V_1 = w \cos \alpha$ = shear force per unit of length; and $M_1 = w\, l \cos \alpha$ = moment force per unit of length.

Where the moment is referred to the center, or gravity, axis, Equations (6), (7), and (8) are reduced to;

$$\Delta_N = \zeta \times \frac{N_1}{E_R} \quad \text{........................}(9)$$

$$\Delta_V = \zeta \times \frac{V_1}{E_R} + \eta \frac{M_1}{E_R\, t} \quad \text{..................}(10)$$

and,

$$\psi = \mu \frac{M_1}{E_R\, t^2} + \eta \frac{V_1}{E_R\, t} \quad \text{...................}(11)$$

in which the values of ζ, η, and μ are obvious by comparison.

For the arch, the moment force about the gravity line is,

$$M_1 = X\,[r_c - r_n \cos \phi_1] + p_e\, r_e\, c \quad \text{..............}(12)$$

and N_1 and V_1 are the same as N and V in Equations (4) and (5), respectively. Inserting these values of N_1, V_1, and M_1, into Equations (6), (7), and (8), and taking the components of Δ_N and Δ_V normal to the crown radius, the foundation yield would allow a crown-section movement of:

$$\Delta_F = \frac{\zeta}{E_R}\,[V_1 \sin \phi_1 - N_1 \cos \phi_1] + \eta \frac{M_1}{E_R\, t} \sin \phi_1$$

$$+ \left[\mu \frac{M_1}{E_R\, t^2} + \eta \frac{V_1}{E_R\, t}\right] [r_n (1 - \cos \phi_1) - c \cos \phi_1] \quad \text{........}(13)$$

or, expanded and re-arranged:

$$\Delta_F = \frac{X}{E_R}\left[\zeta - \eta \frac{r}{t} \cos \phi_1 \sin \phi_1 - \left(\mu \frac{r}{t} \cos \phi_1\right)\left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)\right.$$

$$+ \eta \sin \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \Bigg] + \frac{X\, r_C}{E_R} \left[\eta \sin \phi_1 + \frac{\mu}{t} \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \right]$$

$$+ \frac{p_e\, r_e}{E_R} \left[\eta \frac{c}{t} \sin \phi_1 - \zeta \cos \phi_1 + \mu \frac{c}{t} \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \right] \quad \ldots (14)$$

and,

$$\psi_F = \mu \frac{M_1}{E_R\, t^2} + \eta \frac{V_1}{E_R\, t} = \frac{X}{E_R\, t} \left(\eta \sin \phi_1 - \mu \frac{r}{t} \cos \phi_1 \right)$$

$$+ \frac{X\, r_C\, \mu}{E_R\, t^2} + p_e\, r_e \frac{\mu\, c}{E_R\, t^2} \quad \ldots\ldots (15)$$

Derivation of Variables, K and r_c, Including Yield.—Using the two known end conditions of the crown section, due to symmetry, namely, Equations (16) and (17):

$$\Sigma\, (\Delta_C + \Delta_F) = 0 \quad \ldots\ldots (16)$$

and,

$$\Sigma\, (\psi_C + \psi_F) = 0 \quad \ldots\ldots (17)$$

and substituting for X the product, $K\, p_e\, r_e$, two equations are derived in terms of t, r, ϕ, m, ζ, μ, and η, and the two unknowns, K and r_c. Furthermore, it is assumed that $E_c = E_R$ and that $I_n = I = \frac{1}{12}\, t^3$. Thus:

$$K \left[\sigma \frac{r_n}{t} \sin \phi_1 + \zeta \cos \phi_1 - \eta \sin \phi_1 \frac{c}{t} - \mu \frac{c}{t} \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \right]$$

$$- \Bigg[- 12 \left(\frac{r_n}{t} \right)^3 \left(\sin \phi_1 - \frac{\phi_1}{2} - \tfrac{1}{4} \sin 2\, \phi_1 \right) + \frac{r_n}{t} (1.94\, \phi_1 - 0.47 \sin \phi_1)$$

$$+ \zeta - \eta \sin \phi_1 \frac{r}{t} \cos \phi_1 - \mu \frac{r}{t} \cos \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right)$$

$$+ \eta \sin \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \Bigg] - \frac{r_C}{t} \Bigg[12 \left(\frac{r_n}{t} \right)^2 (\phi_1 - \sin \phi_1)$$

$$+ \eta \sin \phi_1 + \mu \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1 \right) \Bigg] = 0 \quad \ldots\ldots (18)$$

and,

$$\frac{r_C}{t} \left[12 \left(\frac{r_n}{t} \right) \phi_1 + \mu \right] - \left[12 \left(\frac{r_n}{t} \right)^2 - \eta \sin \phi_1 + \mu \frac{r}{t} \cos \phi_1 \right]$$

$$+ K \left[\mu \frac{c}{t} \right] = 0 \quad \ldots\ldots (19)$$

The solution of Equations (18) and (19) simultaneously may be simplified greatly by a convenient grouping of variables such that:

$$Q_1 = \left[-12\left(\frac{r_n}{t}\right)^3 \left(\sin \phi_1 - \frac{\phi_1}{2} - \tfrac{1}{4} \sin 2\,\phi_1\right) + \frac{r_n}{t}\,(1.94\,\phi_1 - 0.47 \sin 2\,\phi_1) + \zeta - \eta \sin \phi_1 \frac{r}{t} \cos \phi_1 - \mu \frac{r}{t} \cos \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right) + \eta \sin \phi_1 \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)\right] \quad \ldots\ldots (20)$$

$$Q_2 = \left[12\left(\frac{r_n}{t}\right)^2 (\phi_1 - \sin \phi_1) + \eta \sin \phi_1 + \mu \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)\right] \quad \ldots (21)$$

$$Q_3 = \left[\sigma \left(\frac{r_n}{t}\right) \sin \phi_1 + \zeta \cos \phi_1 - \eta \frac{c}{t} \sin \phi_1 - \mu \frac{c}{t} \left(\frac{r_n}{t} - \frac{r}{t} \cos \phi_1\right)\right] . (22)$$

$$Q_4 = \left[12\left(\frac{r_n}{t}\right)^2 \sin \phi_1 - \eta \sin \phi_1 + \mu \left(\frac{r}{t}\right) \cos \phi_1 \right] \quad \ldots\ldots (23)$$

$$Q_5 = \left[\mu \left(\frac{c}{t}\right)\right] \quad \ldots\ldots (24)$$

and,

$$Q_6 = \left[12 \left(\frac{r_n}{t}\right) \phi_1 + \mu \right] \quad \ldots\ldots (25)$$

Substituting Equations (20) to (25) in Equations (18) and (19), the following more manageable forms are derived:

$$K\,Q_3 - Q_1 - \frac{r_C}{t}\,Q_2 = 0 \quad \ldots\ldots (26)$$

and,

$$\frac{r_C}{t}\,Q_6 - Q_4 + K\,Q_5 = 0 \quad \ldots\ldots (27)$$

or, solving simultaneously,

$$K = \frac{(Q_3 \times Q_6) + (Q_5 \times Q_2)}{(Q_1 \times Q_6) + (Q_4 \times Q_2)} \quad \ldots\ldots (28)$$

and,

$$\frac{r_C}{t} = \frac{(Q_4 \times Q_3) - (Q_1 \times Q_5)}{(Q_2 \times Q_5) + (Q_3 \times Q_6)} \quad \ldots\ldots (29)$$

Stress Equations in Terms of K and r_C.—Professor Cain has derived the equations of stress for a thick arch in the form:

At the Crown:

$$s_e = \frac{M_c}{I}\,\frac{\frac{t}{2} + c}{r_e}\,r_n - \frac{P_C}{r_e \log_e \left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots (30)$$

and,

$$s_i = -\frac{M_c}{I}\,\frac{\frac{t}{2} - c}{r_i}\,r_n - \frac{P_C}{r_i \log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots\ldots\ldots(31)$$

At the Abutment:

$$s_e = \frac{M_1}{I}\,\frac{\frac{t}{2} + c}{r_e}\,r_n - \frac{P_1}{r_e \log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots\ldots\ldots(32)$$

and,

$$s_i = -\frac{M_1}{I}\,\frac{\frac{t}{2} - c}{r_i}\,r_n - \frac{P_1}{r_i \log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots\ldots\ldots(33)$$

Equations (30) to (33) may be converted into a form convenient for computing and platting by noting that,

$$\log_e\left(\frac{r_e}{r_i}\right) = \frac{t}{r_n} \text{ (approximately)} \quad \ldots\ldots\ldots\ldots(34)$$

$$P = p_e\,r_e - X\cos\phi \quad \ldots\ldots\ldots\ldots(35)$$

$$M = X(r_C - r_n\cos\phi) \quad \ldots\ldots\ldots\ldots(36)$$

$$X = p_e\,r_e\,K \quad \ldots\ldots\ldots\ldots(37)$$

$$12\,\frac{r_n}{t}\left(\frac{1}{2} + \frac{c}{t}\right) = n_e \quad \ldots\ldots\ldots\ldots(38)$$

and,

$$12\,\frac{r_n}{t}\left(\frac{1}{2} - \frac{c}{t}\right) = n_i \quad \ldots\ldots\ldots\ldots(39)$$

Making the necessary substitutions, Equations (30) to (33) reduce to:

At the Crown:

$$\frac{s_e}{p_e} = n_e\left(K\,\frac{r_C}{t}\right) + \frac{r_n}{t}\,(1 - n_e)\,K - \frac{r_n}{t} \quad \ldots\ldots\ldots\ldots(40)$$

and,

$$\frac{s_i}{p_e} = -\frac{r_e}{r_i} \times n_i\left(K\,\frac{r_C}{t}\right) + \frac{r_e}{r_i} \times \frac{r_n}{t}\,(1 + n_i)\,K - \frac{r_e}{r_i} \times \frac{r_n}{t} \quad \ldots(41)$$

At the Abutment:

$$\frac{s_e}{p_e} = n_e\left(K\,\frac{r_C}{t}\right) + \frac{r_n}{t}\,(1 - n_e)\,K\cos\phi_1 - \frac{r_n}{t} \quad \ldots\ldots\ldots(42)$$

and,

$$\frac{s_i}{p_e} = -\frac{r_e}{r_i} \times n_i\left(K\frac{r_C}{t}\right) + \frac{r_e}{r_i} \times \frac{r_n}{t}(1 + n_i)\,K\cos\phi_1 - \frac{r_n}{t} \times \frac{r_e}{r_i} \quad \ldots (43)$$

Effect of Variations in E_R and m

The equations introduced in this paper correspond with comparable equations derived by Cain with the added advantages that the effect of yielding abutments and a finite value of Poisson's ratio, are taken into consideration. To simplify the use of these equations a set of graphs is necessary[6a], similar to those presented by Fowler in 1927. Figs. 3, 4, 5, and 6 are examples of such curves. These graphs are based on the following conditions, assumptions, or simplifications: (*a*) The arch is circular and of uniform thickness; (*b*) $I_n \doteq \frac{1}{12}t^3$; (*c*) $\frac{E_v}{E_c} \times 1.2 = 2.88$ (in which, E_v = the modulus of elasticity in shear); (*d*) the trace of the abutment plane is normal to the arch rib; (*e*) the loading over the entire arch is uniform; (*f*) the stress due to vertical loads is zero; (*g*) $E_c = E_R$; (*h*) $m = 8$; and (*i*) tensile stresses are positive.

As the values of E_R and m selected for the computations are open to conjecture, the effect of a reasonable variation from the chosen values is to be examined. The result of solving a specific example, in which the effect of abutment yield is quite appreciable (since the ratio, $\frac{t}{r}$, is large), is shown in Table 1(*a*), where the stresses when $E_R = 75\%\ E_c$ and $E_R = 125\%\ E_c$ are compared with those obtained when it is assumed that $E_R = E_c$. In this example, $t = 30$ ft; $r = 60$ ft; $2\,\phi_1 = 120°$; and $\frac{h}{t} = 2$. The values in

TABLE 1.—Stresses (in Pounds per Square Inch) for Unit Head in an Arch with Yielding Abutments

Variant	Crown		Abutment	
	Extrados	Intrados	Extrados	Intrados
(*a*) Comparison of Stresses for Various Values of E_R				
$E_R = E_c$	−1.35	−0.14	−0.07	−2.18
$E_R = \frac{3}{4} E_c$	−1.34	−0.09	−0.15	−2.08
$E_R = 1\frac{1}{4} E_c$	−1.28	−0.16	−0.03	−2.25
(*b*) Comparison of Stresses for Various Values of Poisson's Ratio				
Fixed abutments; $m = 8$	−1.18	−0.13	+0.41	−2.77
Yielding abutments; $m = 8$	−1.35	−0.14	−0.07	−2.18
Yielding abutments; $m = 5$	−1.29	−0.18	−0.10	−2.16

[6a] A complete lithographic set of these curves suitable for the designer may be purchased at a cost of 50 cents per copy from the Secretary of the Society.

Table 1(*a*) are derived with the assistance of Figs. 3(*a*), 4(*a*), 5(*a*), and 6(*a*), by first entering the charts from the bottom at the given value of $\phi = \frac{120}{2} = 60$ degrees. Then continue vertically to an intersection with the proper curve of $\frac{t}{r} = \frac{30}{60} = 0.5$. The stresses for unit head on the horizontal lines through these intersections may then be read.

The greatest variation from the assumed E_R amounts to only approximately 0.10 lb per sq in. for unit head, which, for a 100-ft head, would change the stresses by only 10 lb per sq in.

In Table 1(*b*), the stresses when $m = 5$ are compared with those when $m = 8$, the value adopted for the curves. As in the previous example, $\frac{t}{r} = 0.5$, $2\ \phi_1 = 120°$, and $\frac{h}{t} = 2$, and the stresses are derived by means of Fig. 3(*a*), 4(*a*), 5(*a*), and 6(*a*). Furthermore, in order to show the comparative effect of including abutment yield, the stresses obtained with the assumption of fixed ends, as in Fowler's curves, are included.

Table 1(*b*) indicates that the stresses are quite insensitive to fairly large variations in the value of *m*. For this typical example, furthermore, the yielding abutment calculations show a more equable distribution of stress between crown and abutment by increasing the critical crown extrados stresses and reducing the critical abutment intrados stresses. In this particular example, tension at the abutment extrados found by the fixed-end assumption is substituted by a small compression upon introducing the yielding foundations.

Example to Demonstrate the Use of the Curves

As a typical case, assume that $\frac{h}{t} = 7$; thickness of arch, $t = 40$ ft; central radius $= 100$ ft; $2\ \phi_1 = 65°$; head of water $= 150$ ft; and that the arch carries 75% of the load. This problem is solved by determining stresses separately from Figs. 3, 4, 5, and 6, for $\frac{h}{t} = 2$ and $\frac{h}{t} = 12$. The unit-head stress for $\frac{h}{t} = 7$ is then found by interpolation. The final process is illustrated by Table 2. To obtain the arch stresses from the unit-head stress, values of unit stresses in Table 2 are multiplied by 150×0.75.

TABLE 2.—Stresses (in Pounds per Square Inch), in an Arch with Yielding Abutments

Section	Unit-Head Stress for the Following Ratios of $\frac{h}{t}$:			Arch stress
	2	12	7	
Crown extrados	−1.31	−1.37	−1.34	−151
Crown intrados	+0.72	+0.88	+0.80	+ 90.0
Abutment extrados	+0.35	+0.37	+0.36	+ 40.5
Abutment intrados	−1.77	−1.73	−1.75	−197

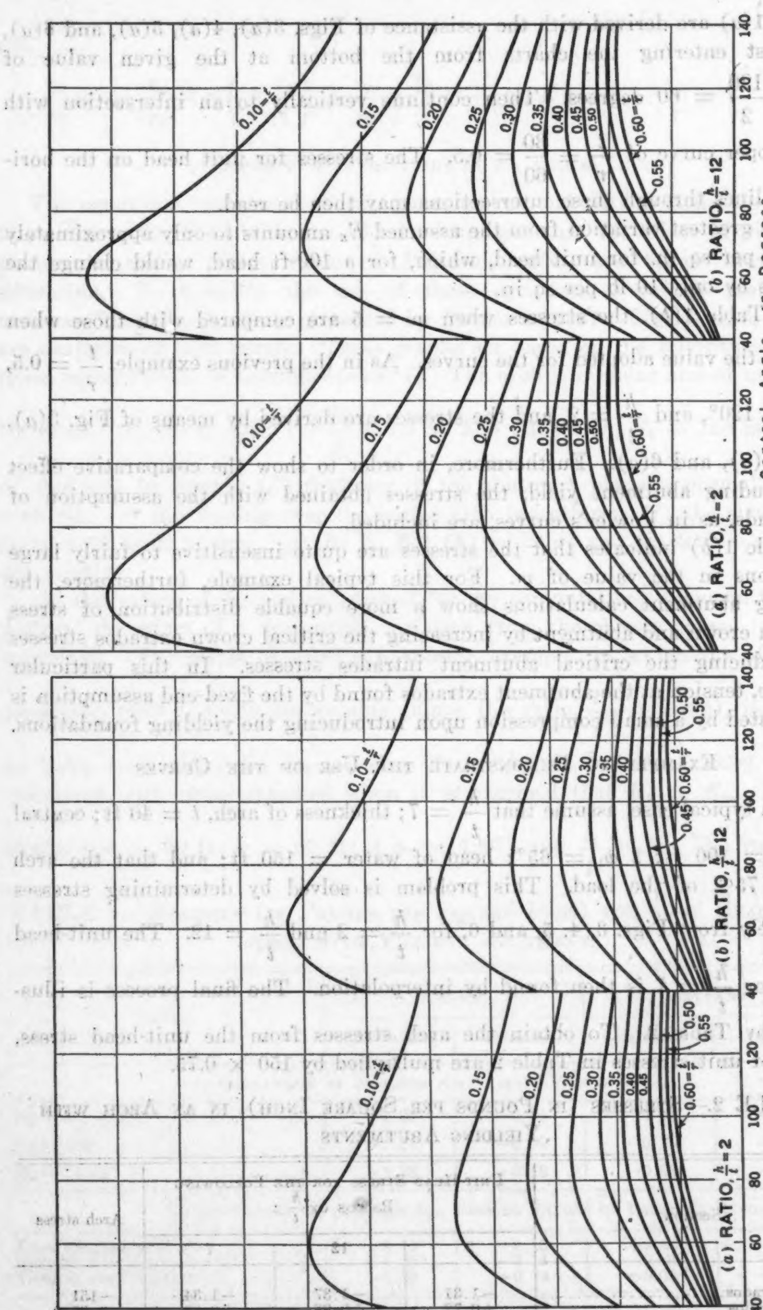

FIG. 3.—CROWN STRESSES, EXTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

FIG. 4.—ABATEMENT STRESSES, INTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

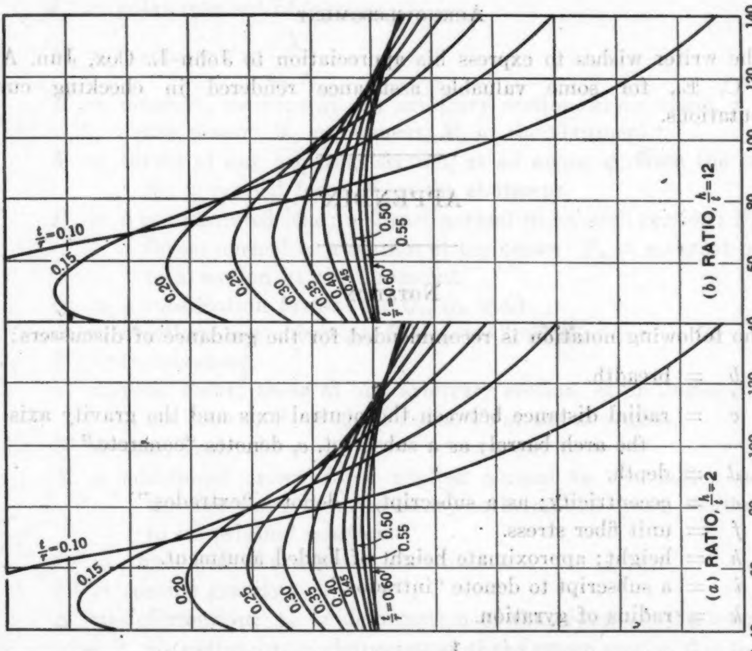

FIG. 6.—ABUTMENT STRESSES, EXTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

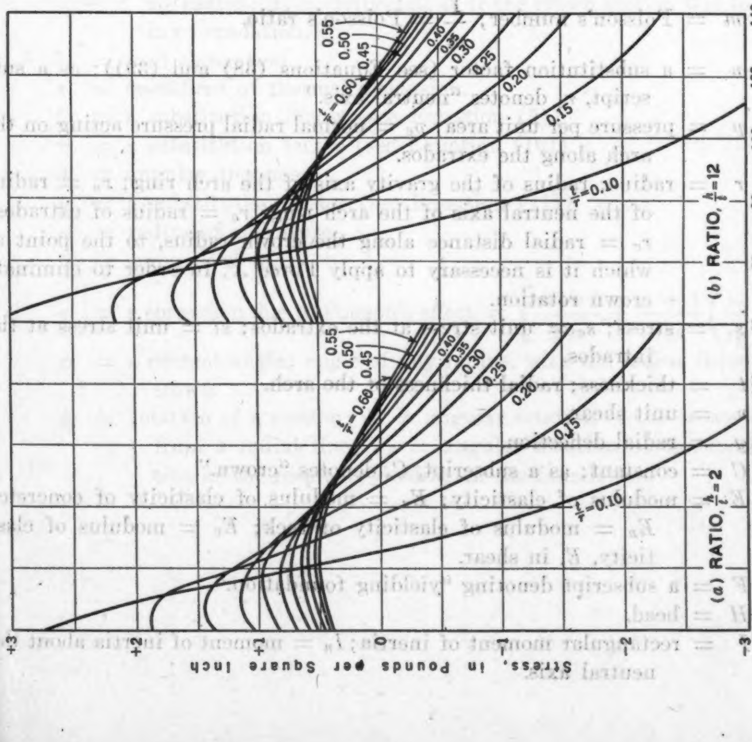

FIG. 5.—CROWN STRESSES, INTRADOS, FOR THE CASE OF YIELDING FOUNDATIONS.

Acknowledgment

The writer wishes to express his appreciation to John L. Cox, Jun. Am. Soc. C. E., for some valuable assistance rendered in checking curve computations.

APPENDIX

Notation

The following notation is recommended for the guidance of discussers:

b = breadth.

c = radial distance between the neutral axis and the gravity axis of the arch barrel; as a subscript, c, denotes "concrete."

d = depth.

e = eccentricity; as a subscript, e, denotes "extrados."

f = unit fiber stress.

h = height; approximate height of loaded abutment.

i = a subscript to denote "intrados."

k = radius of gyration.

m = Poisson's number; $\frac{1}{m}$ = Poisson's ratio.

n = a substitution factor (see Equations (38) and (39)); as a subscript, n, denotes "neutral axis."

p = pressure per unit area; p_e = normal radial pressure acting on the arch along the extrados.

r = radius; radius of the gravity axis of the arch ring; r_n = radius of the neutral axis of the arch ring; r_e = radius of extrados; r_C = radial distance along the crown radius, to the point at which it is necessary to apply Force X, in order to eliminate crown rotation.

s = stress; s_e = unit stress at the extrados; s_i = unit stress at the intrados.

t = thickness; radial thickness of the arch.

v = unit shear.

y = radial deflection.

C = constant; as a subscript, C, denotes "crown."

E = modulus of elasticity; E_c = modulus of elasticity of concrete; E_R = modulus of elasticity of rock; E_v = modulus of elasticity, E, in shear.

F = a subscript denoting "yielding foundation."

H = head.

I = rectangular moment of inertia; I_n = moment of inertia about the neutral axis.

J = polar moment of inertia.

K = a constant $= \frac{X}{p_e \, r_e}$.

M = moment; moment at any arbitrary section, at an angle, ϕ, from the crown; M_1 = moment, M, at the abutment.

N = thrust at any arbitrary section, at an angle, ϕ, from the crown; N_1 = normal force, N, at the abutment.

P = a concentrated load; a thrust normal to an arch section; P_C = a thrust normal to a section at the crown; P_1 = a thrust normal to a section at the abutment.

Q = a substitution factor (see Q_1, Q_2, etc.).

R = reaction; as a subscript, R, denotes "rock."

T = temperature.

V = total shear; shear at any arbitrary section, at an angle, ϕ, from the crown; V_1 = shear force, V, at the abutment.

W = total load.

X = additional crown force applied normal to the radius passing through the crown, necessary to bring the crown section back to its original position.

α = angle with the normal, or radial.

ξ = specific gravity;

Δ = deformation; Δ_C = deformation of the crown section normal to its radius; Δ_F = deformation of the crown section due to yielding foundation.

δ = unit elongation.

ε = coefficient of thermal expansion.

ζ = a substitution factor (see Equation (9)).

η = a substitution factor (see Equation (10)).

θ = angular distance.

μ = a substitution factor (see Equation (11)).

ρ = radius of curvature.

Σ = summation of.

σ = a correction due to Poisson's effect $= \left[\frac{m-1}{2\,m} + \frac{m+1}{2\,m}\left(\frac{r_i}{r_n}\right)^2\right]\frac{r_e}{r_i}$.

ϕ = a central angle; angle of any radius, with the radius through the crown.

ψ = rotation of a section; ψ_C = angular rotation of the crown section from a radial line; ψ_F = angular rotation of the crown section as the result of yielding in the foundation.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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PAPERS

HYDRAULIC LABORATORY RESULTS AND THEIR VERIFICATION IN NATURE

By Herbert D. Vogel,[1] Assoc. M. Am. Soc. C. E.

Synopsis

Data contained in this paper were assembled in 1933, added to in January, 1934, and again augmented in the summer of the latter year. Nevertheless, it is quite certain that the intervening months have produced many new facts corroborating the information set forth herein, and while it was not originally intended to present anything like a complete inventory of cases, still it would appear desirable to bring out all new facts in the order of their determination. From this standpoint it would seem that the proper purpose of the paper is to discuss in general the principles of verification of hydraulic model results, to present a number of concrete examples, and to open the way for subsequent discussions. Since the subject can never become closed, it would be presumptious to claim more than a "scratching of the surface" in the present instance. In spite of this, the hope is held that the following pages will serve to assure the profession that efforts are being made to verify the reliability of results obtained by hydraulic model experimentation.

Introduction

What reliance can be placed by field engineers on the solutions obtained from model studies and what proof is there of the statement that a river will re-act to treatment the same as its miniature in a laboratory? These are the questions most frequently asked of those engaged in the task of wresting river secrets from small-scale models, and—strangely enough—they are the most difficult to answer convincingly. Not that insufficient work has been done in laboratories to afford proof of one kind or another, but that positive data and positive facts are so few in the general case as to make positive

Note.—Presented at the meeting of the Waterways Division, New York, N. Y., January 18, 1934. Discussion on this paper will be closed in April, 1935, *Proceedings*.

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proof next to impossible. It is noteworthy that of the topics assigned for discussion at the XVth International Congress of Navigation (Venice, 1931), one was:

"The study of hydrotechnical questions by means of laboratory researches on reduced scale models. Comparison of the results of such researches with those of direct observations of the natural phenomena, with a view to ascertaining how far the law of similitude is true", * * *.

And that of all the papers submitted only one or two authors showed the hardihood to come down to definite cases of comparison. Furthermore, that German paper which most boldly attacked the subject offered as one of its strongest items of evidence the experiments for determining the spillway discharge of Keokuk Dam, in Iowa, reported in 1929 by Albion Davis and the late Floyd A. Nagler, Members, Am. Soc. C. E.[3] It was shown, as a result of tests on a 1:11 scale model and the actual structure, that there was close agreement between discharge coefficients and other hydraulic functions in model and Nature.

There are many reasons for the difficulties in citing actual instances of complete verification, in Nature, of model findings. Were this not true it would follow either that model tests are entirely unreliable, or that no attempts have been made to present evidence of verifications. The latter assumption having been shown to be in error, and the best authorities having agreed that beneficial results may be obtained from a well conducted model study, it remains to cite a few of the more common reasons for the difficulties that have been encountered.

To begin with, only the more difficult problems are submitted to the laboratory for analysis by experimental means. This is quite natural, of course, since there is no reason for making model studies of problems which present no doubtful elements, or for which there are already satisfactorily established practices. In the second place, most of the problems submitted to the laboratory are accompanied by schedules of detailed proposed plans, each of which is to be tested separately in the model. As a result of the experiment it may be necessary to modify some of the plans in order to find one or more that will produce the desired results. By the time any of these plans has been accepted and instituted in the field it has almost invariably been somewhat further modified to suit the character of changes in local conditions that have taken place in the mean time. Naturally, the difficulty of comparing effects of the work finally installed with those indicated as being most suitable by the model increases with the degree of the modification.

Thirdly, the time required in Nature for structures to become effective differs greatly with the type of the works involved and the character of local conditions. Works, for instance, which are designed to produce changes in the river bed by inducing deposition of silt would be more quickly effective on the heavily silt-laden streams of the West and Southwest than on such comparatively clear rivers as the Ohio, or the Upper Mississippi. If properly located, spur-dikes installed to increase depth of channel at low water by

[3] *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 777.

their scouring action, become effective relatively soon in rivers having mud or sand bottoms, whereas in localities in which there are deposits of gravel such dikes may achieve the desired results only at a much later date.

To obtain the results, in Nature, indicated by a model study, the improvement works should be installed as soon as possible. This is particularly important when the river, or the part involved, is changing actively. In a large alluvial river, such as the Lower Mississippi, delay of one or more working seasons in instituting a plan may be attended by such extensive changes in the river as to affect the success of the proposed plan materially.

In the case of the United States Waterways Experiment Station, experimental work was begun in January, 1931, and the first experiments were not completed until from six months to a year later. A similar period necessarily elapsed before any works, tested during these experiments, could be installed. Consequently, sufficient time has not elapsed to demonstrate conclusive results in every case. However, proofs are not entirely lacking, and a number of cases in which field checks on model results have been obtained, either partly or completely, will be discussed in this paper.

Verification of Model Results

General.—The present Mississippi River navigation program (adopted January 21, 1927) provides for the procurement of a navigable channel 9 ft deep and 300 ft wide, for all stages of the river, between St. Louis, Mo., and Baton Rouge, La. In this program the principal reliance has been placed on regulatory works, supplemented, where necessary, by channel dredging. Standard practice has been generally followed in the design and location of these works, but in more and more instances, and especially when unusual or baffling conditions are presented, the problems are referred to the U. S. Waterways Experiment Station for study and recommendations.

A technique was quickly evolved for the study of this particular type of problem, although new methods, involving movable-bed models, were required, and special questions, such as those of limiting slopes and of character of sand necessary for the sand bed, required almost immediate answers. Where exigencies of the situation demanded, it was found possible to design and construct a model within less than a month after the receipt of survey data from the field, and to begin the release of experimental data immediately thereafter. This is seldom desirable, however, and with less haste necessary, experiments proceed at a more leisurely pace, some of the studies having been extended for as long as 1½ years.

In certain instances, models have been operated concurrently with the installation of improvement works in the prototype. In such cases, the various construction details, together with any natural changes, have been reproduced as they progressed in Nature, and findings have been forwarded to the field engineers either by mail or by radio.

Field representatives of the various districts for which models have been built are frequent visitors to the Laboratory for the purpose of inspecting the progress of the experiments. Only rarely do these visiting engineers fail

to point to some phenomena in the model which are verified by existing conditions in Nature. It may, perhaps, be the formation of a sand bar or a gravel bar, an eddy, a caving bank, or scour in a certain vicinity; or, it may be the development of the general configuration of the bed of the model which bears such a close resemblance to that of the natural reach that it becomes difficult to distinguish between maps of the two. It is noticeable that these engineers nearly always find in the model a simulation of those characteristics of the river with which they are most familiar. This fact lends an added value to such testimonials. Numerous instances are on record in which developments in the natural stream have been predicted from the model studies many months in advance of their actual occurrence.

Hydraulic models have two general functions: First, by their aid a more exhaustive and detailed estimate and evaluation may be made of existing conditions in the prototype, and, as a result of the information thus secured, plans for improvement may be conceived with greater likelihood of effectiveness and economy than would otherwise be possible; and, second, the merit of these plans may then be determined in the model itself. Before a model can be relied upon, confidently, to give such important information, its capacity to simulate known past and present conditions must first be demonstrated. In laboratory parlance, the ascertainment of such facts is termed the "verification of the model".

Theory of Verification.—The first step in the vertification of a model is a manual one, that of checking it dimensionally to insure that geometric similarity between the model and the corresponding region in Nature has been obtained. Following this preliminary check, to which every model is subjected, it remains to prove that hydraulic similarity exists, or that results of tests conducted on the model may be used as the basis of designing regulatory works for the full-sized river, harbor, etc. In general, this hydraulic similarity can be verified in any one or all of several manners.

Verification (1).—In the case of a movable-bed model, checks can be made by subjecting it to "moulded-bed" or to "flat-bed" verification runs. If two or more surveys of the vicinity are available, the first method is used. The movable-bed part of the model is moulded to conform to known conditions of the earlier date; the model is subjected to several cycles of flow at varying stages, representing average hydrographs in Nature; it is considered trustworthy only if its capability to reproduce the known conditions of the later date can be demonstrated.

In instances in which field information taken at different dates is not available, the second method is used. The movable-bed part of the model is moulded flat; water is passed through the model at varying stages in simulation of average conditions in Nature; and the test is completed when the model has moulded its bed so that it conforms to the hydrographic conditions in Nature at the time of the survey from which the model was constructed. The reasoning behind such a test lies in the fact that a stream adjusts the configurations of its bed in accordance with its hydraulic regimen and the alignment of its banks. If the dimensional check of the model

proves its geometric similarity to the prototype and, if hydraulic similarity exists, the model should mould its own bed in simulation of Nature. All the movable-bed models studied at the U. S. Waterways Experiment Station are subjected to one or the other of these verification tests, and many of them are given both checks.

Verification (2).—The more practical application of this type of check is obtained when a comparison can be made between conditions as they develop in a stream after the completion of a model study and the conditions which the model indicated would develop. This type of field verification can be made in the case of: (*a*) Movable-bed models by comparing bed configurations in streams with the configurations predicted from model studies; (*b*) fixed-bed river model (when it is desired to determine the changes in gauge heights, velocity distributions, etc., resulting from cut-offs or other regulatory work) by a comparison of model predictions with actual developments in Nature; and (*c*) models of spillways, turbines, etc., by similar comparisons.

The work of the U. S. Waterways Experiment Station is too new to have yielded many conclusive verifications of this kind. However, several such comparisons are available, and some of them will be discussed in detail. It should be pointed out that these two types of verifications are fundamentally the same. In each case, a comparison is made between the development of the model and the corresponding development of the prototype. The only difference lies in the time at which the model study is made—whether at the end of the period of development, or at the beginning.

Verification (3).—A third check may be obtained if two or more models to different scales have been used for the study of the same problem. A transference of the results of the small-scale model (by means of the laws of similitude) to the scale of the larger model provides a check on the accuracy of the results of both models. If the two are in fair agreement, it is reasonable to assume the next step in the extrapolation of the model data to the full scale of the prototype. Several examples of each type of verification are described herein.

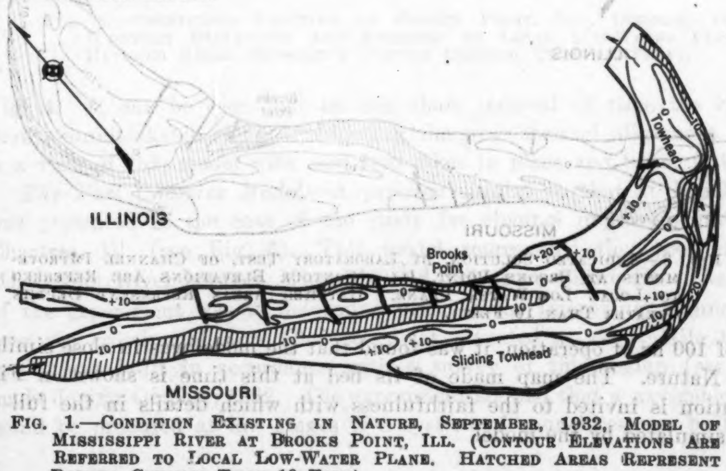

FIG. 1.—CONDITION EXISTING IN NATURE, SEPTEMBER, 1932, MODEL OF MISSISSIPPI RIVER AT BROOKS POINT, ILL. (CONTOUR ELEVATIONS ARE REFERRED TO LOCAL LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 10 FEET).

The Brooks Point Model.—The model of Brooks Point, Ill. (Fig. 1), was built in simulation of a 10-mile stretch of the Mississippi River lying between Mile 20 and Mile 30 above Cairo, Ill. The purpose of the model study was to devise a means for improving the depth of the navigation channel over

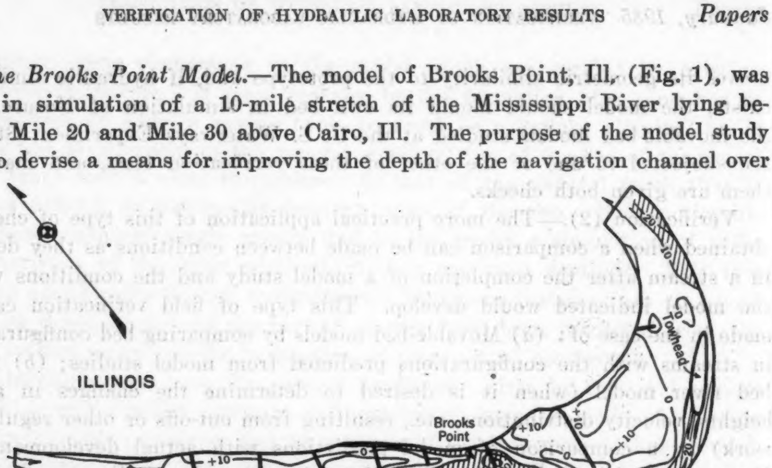

FIG. 2.—FLAT BED VERIFICATION RUN, MODEL OF MISSISSIPPI RIVER AT BROOKS POINT, ILL. (CONTOUR ELEVATIONS ARE REFERRED TO LOCAL LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 10 FEET).

the crossing at that point during low-water stages. Only one general survey of the region was available to the Laboratory. This was made in September, 1932, and the model was constructed from the data thereof. Fig. 1 shows the conditions in the prototype as revealed by this survey. Lacking any other survey data, the model was subjected to the flat-bed verification test. At the

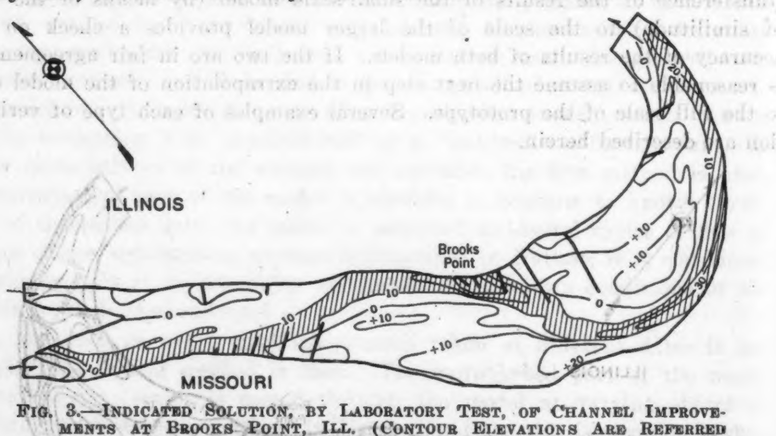

FIG. 3.—INDICATED SOLUTION, BY LABORATORY TEST, OF CHANNEL IMPROVEMENTS AT BROOKS POINT, ILL. (CONTOUR ELEVATIONS ARE REFERRED TO LOCAL LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 10 FEET).

end of 100 hr of operation, it was found that the model was in close similarity with Nature. The map made of its bed at this time is shown in Fig. 2. Attention is invited to the faithfulness with which details in the full scale were simulated by the model.

While the experiments on the model of Brooks Point were in progress the Station was visited several times by the engineers in charge of the channel improvement works in the vicinity represented by the model. They were unanimous in voicing their opinion that the model simulated to a marked degree the conditions of Nature.

This model was successful in demonstrating several practicable plans for the improvement of the channel at Brooks Point. One of these, which provided for the construction of five spur-dikes on the right bank just above the shoal crossing, and the removal of three existing dikes on the left bank at a point just opposite, was selected. The details of this plan are shown in Fig. 3. In order to preserve the best conditions for navigation, the removal of the three dikes was conducted concurrently with the construction of the five new dikes. The new dikes were completed in August, 1933, but only sections of the old dikes had been removed at this time. A survey of this vicinity, made about two months later (October, 1933), is shown in

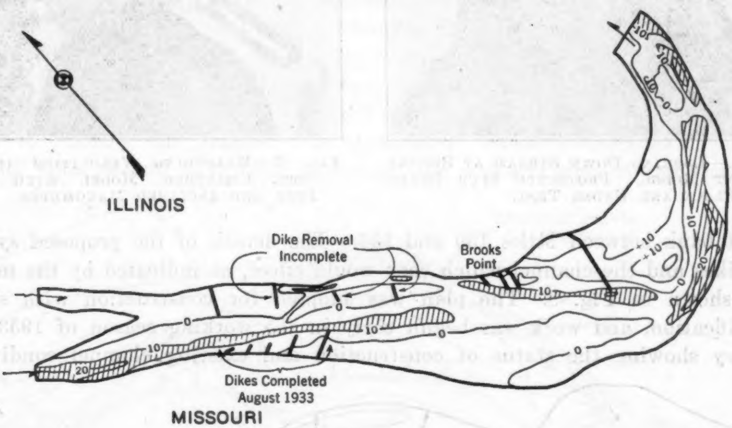

FIG. 4.—CONDITIONS EXISTING AT BROOKS POINT, ILL., OCTOBER, 1933 (CONTOUR ELEVATIONS ARE REFERRED TO LOCAL LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 10 FEET).

Fig. 4. It can be seen that in this short interval of time the river has shown unmistakable signs of adopting the new channel alignment. Fig. 5 is a view of this model with new spur-dikes in place and being tested.

The Fort Chartres Model.—A problem similar to that at Brooks Point was presented in the case of the study for channel improvements at Fort Chartres, Ill. (see Fig. 6). This model represented that section of the Mississippi River between Miles 120 and 137 above Cairo, Ill. The purpose of the experiment was to determine methods for improving channel depths over the crossing at Fort Chartres, East, situated at about Mile 131. The model was built in accordance with a survey of this region (see Fig. 7) made during October, 1932. The experiment showed that a navigable channel could be procured at low stages if a system of spur-dikes was built on the

FIG. 5.—LOOKING DOWN STREAM AT BROOKS POINT MODEL. PROJECTED SPUR DIKES IN PLACE AND UNDER TEST.

FIG. 6—MEASURING VELOCITIES IN THE FORT CHARTRES MODEL WITH PITOT TUBE AND INCLINED MANOMETER.

right bank between Miles 130 and 133. The details of the proposed system of dikes and the channel which they would effect, as indicated by the model, are shown in Fig. 8. The plan was adopted for construction with slight modification, and work was begun early in the working season of 1933. A survey showing the status of construction and existing channel conditions

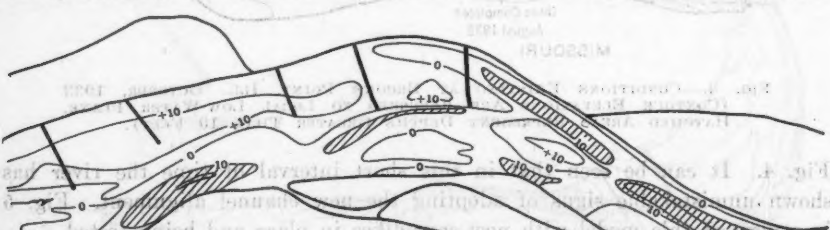

FIG 7.—CONDITION OF RIVER BED AT FORT CHARTRES, ILL., AS SHOWN BY SURVEY OF OCTOBER, 1932.

was made of this region in March, 1934, and is shown in Fig. 9. Although sufficient time has not elapsed for the dikes to become completely effective, it will be noted that they have already caused a relocation of the channel along the desired alignment and that project depths are being obtained along this trace.

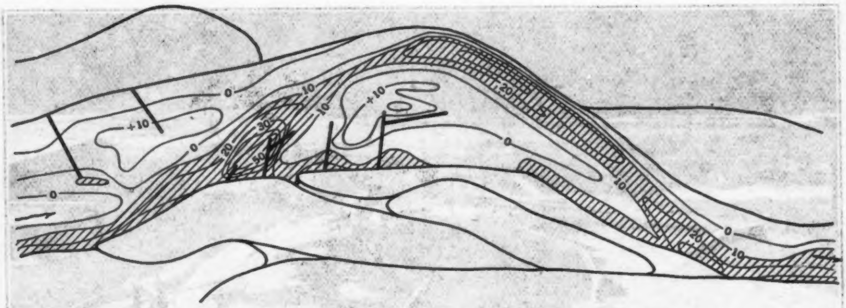

FIG. 8.—PLAN FOR IMPROVEMENT OF CHANNEL AT FORT CHARTRES, ILL., INDICATED BY MODEL TEST.

The Point Pleasant Model.—The Point Pleasant (Mo.), reach of the Mississippi River lies about 80 miles below Cairo, Ill. It is one of several localities in which the low-water channel is characterized by excessive width and consequent deficient depth. A system containing three permeable pile-dikes with a combined length of 8 232 lin ft was completed during 1932 for the improvement of depths over the crossing-bar. Surveys made during August, 1932, showed no improvement in the channel depth opposite the dikes and their ability to improve the channel was questioned. Some authorities thought that the dikes should have been placed on the opposite side of the stream,

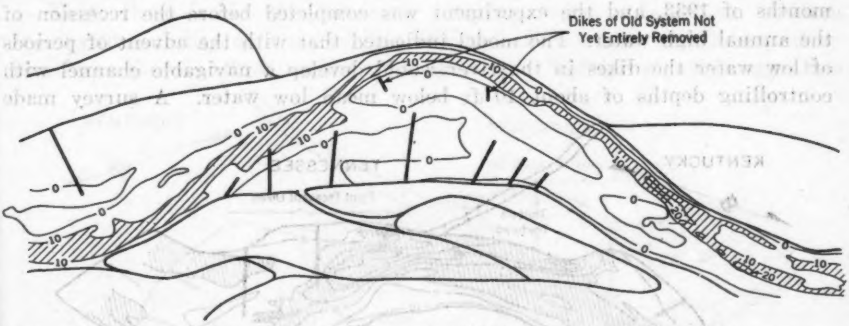

FIG. 9.—PLAN UNDER EXECUTION IN NATURE, CHANNEL AT FORT CHARTRES, ILL., SHOWING CONDITION IN MARCH, 1924.

and advocated their removal. At the direction of the President of the Mississippi River Commission, a model was constructed which simulated an 8-mile stretch of the river, extending from Mile 75 to Mile 83, with the Point Pleasant dikes near the center (see Fig. 10). After the model had been properly verified, it was moulded to conform to conditions shown by the latest field survey, which was made during August, 1932 (see Fig. 11). The findings of the model not only substantiated conditions in the field, but indicated that the Point Pleasant dikes were properly designed and located, and eventually would cause the channel to scour to navigable depths (see Fig. 12).

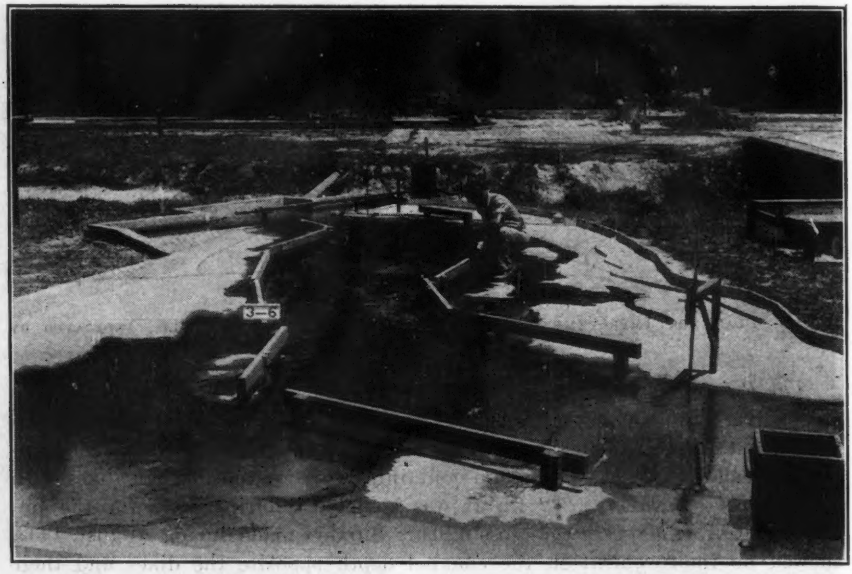

FIG. 10.—VIEW OF POINT PLEASANT MODEL.

Construction of the Point Pleasant model was begun during the first months of 1933, and the experiment was completed before the recession of the annual high water. The model indicated that with the advent of periods of low water the dikes in the river would develop a navigable channel with controlling depths of about 10 ft below mean low water. A survey made

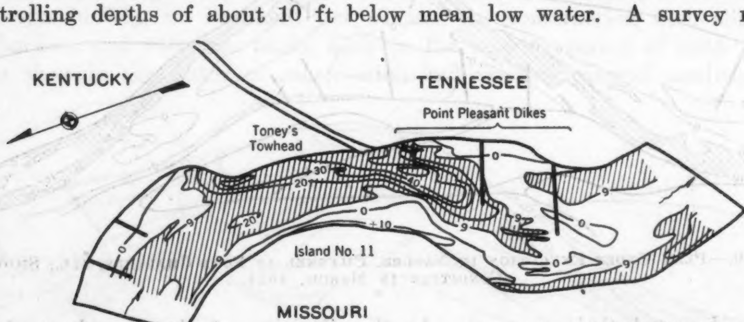

FIG. 11.—CONDITION AT THE POINT PLEASANT, ILL., REACH, IN AUGUST, 1932 (CONTOUR ELEVATIONS ARE REFERRED TO MEAN LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 9 FEET BELOW MEAN LOW-WATER).

by the Area Engineer during July, 1933 (see Fig. 13), showed the crossing-bars to be shaped similarly to those indicated by the model, and revealed depths of 10 ft almost entirely across the critical section. During the next month a continuous channel, 10 ft deep, would probably have developed,

as scouring occurs on the crossings during low water. This would have given a complete verification of the model predictions. In order that there should be no jeopardy to navigation, however, some dredging was done at that time (July). Thereafter, the Area Engineer reported under date of October 27, 1933, that the channel gave no trouble whatever. In fact, as of that

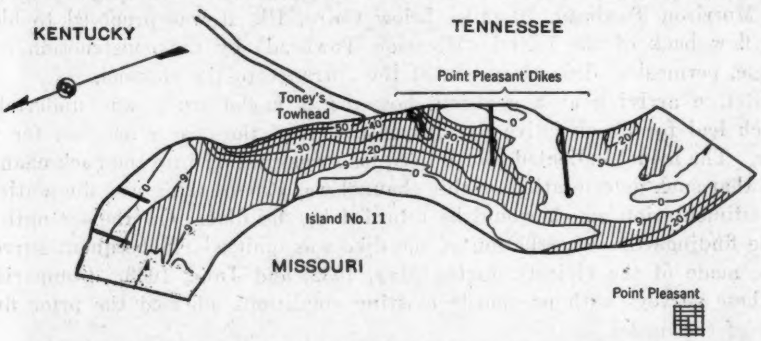

FIG. 12.—LABORATORY TEST OF POINT PLEASANT, ILL., MODEL INDICATED EVENTUAL EFFECTIVENESS OF STRUCTURES IN ABOUT ONE YEAR (CONTOUR ELEVATIONS ARE REFERRED TO MEAN LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 9 FEET BELOW MEAN LOW-WATER).

date, it had scoured approximately 3 ft. deeper. This was the first time in years, that the crossing was not dredged several times.

Thus, there is—in the foregoing case—an instance in which, through the instrumentality of a model, the eventual effectiveness of existing structures was shown. Such results may be termed "ratifying". The aggregate

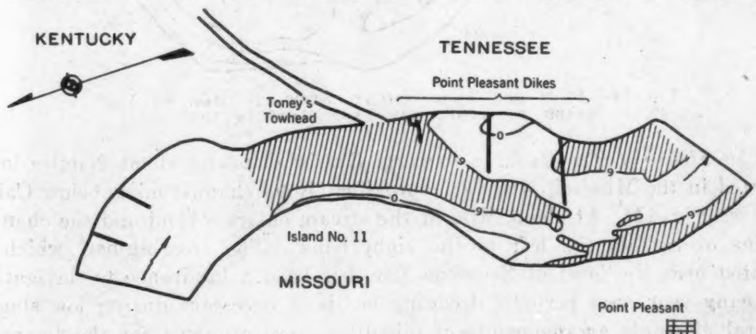

FIG. 13.—CONDITION OF CHANNEL IN POINT PLEASANT REACH, IN JULY, 1933, ELEVEN MONTHS AFTER ORIGINAL SURVEY (CONTOUR INTERVALS ARE REFERRED TO MEAN LOW-WATER PLANE. HATCHED AREAS REPRESENT DEPTHS GREATER THAN 9 FEET BELOW MEAN LOW-WATER).

cost of the Point Pleasant dikes, at an average cost per linear foot of between \$30 and \$35, was approximately \$265 000. Since it is frequently as expensive to remove dikes from a stream as to build them, it would seem that results which lead to such economies are of as great value as those which indicate new plans of attack.

The Morrison Towhead Model.—Another instance in which a model gave so-called "gratifying" results, is that of the model of the Morrison Towhead. The contraction works program on the Lower Mississippi River provides for the closure, where indicated, of certain secondary or back channels, in order to increase the volume of flow in the main channel during low stages. At Morrison Towhead, 70 miles below Cairo, Ill., it was proposed to block the flow back of the island (Morrison Towhead) by the construction of a single, permeable, deflecting dike at the entrance to the channel.

Before arriving at a decision, however, a model study was undertaken which had for its objective the determination of the proper location for the dike. The model indicated that no bed-load was passing into the back channel and that such deterioration as this channel was experiencing was due entirely to sedimentation which would be retarded by the dike. On the strength of these findings the construction of the dike was omitted. Subsequent surveys were made of the vicinity during May, 1932, and June, 1933. Comparison of these surveys with previously existing conditions affirmed the prior findings of the model.

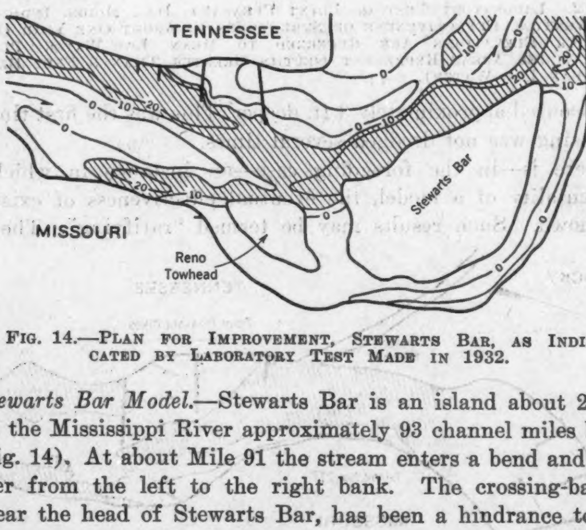

FIG. 14.—PLAN FOR IMPROVEMENT, STEWARTS BAR, AS INDICATED BY LABORATORY TEST MADE IN 1932.

The Stewarts Bar Model.—Stewarts Bar is an island about 2 miles long, situated in the Mississippi River approximately 93 channel miles below Cairo, Ill. (see Fig. 14). At about Mile 91 the stream enters a bend and the channel swings over from the left to the right bank. The crossing-bar, which is situated near the head of Stewarts Bar, has been a hindrance to navigation for many years and periodic dredging has been necessary during low stages. Several alternate arrangements of pile-dikes were proposed for the improvement of this reach. The situation was complicated by the fact that over-contraction might re-open the secondary channel behind Reno Towhead and Stewarts Bar and so reduce the flow in the main channel as to cause a deterioration rather than an improvement of navigation conditions. The problem was referred to the Laboratory for study and report.

A movable-bed model, having a horizontal scale of 1:1 000 and a vertical scale of 1:100, was constructed and tests were made during 1932 of each of the proposed plans. In these tests the engineers of the Laboratory considered not only various locations and lengths for the dikes, but dikes of

different height and inclination with respect to the flow of the current. A report was rendered which indicated that any one of several of the proposed plans would be effective in producing the desired results.

One of the plans which indicated satisfactory results provided for the construction of four spur-dikes on the left bank between Miles 90.8 and 92.0 (see Fig. 14). With some slight modifications this plan was adopted, approved for construction, and the dikes were completed in December, 1932.

To ascertain what effect the dikes were having, a re-survey was made of the Stewarts Bar Crossing during November, 1933 (see Fig. 15). An examination of the results of this survey shows that the dikes had already secured a channel of the desired depth. Comparing Figs. 14 and 15 a remarkable agreement will be found. In this connection, it should be borne in mind that the full effects of a system of dikes are seldom realized during the first season.

Under date of October 27, 1933, the District U. S. Engineer's Office, at Memphis, Tenn., reported that no dredging was done in this reach during the low-water season, 1933, because, for the first time in years, the channel

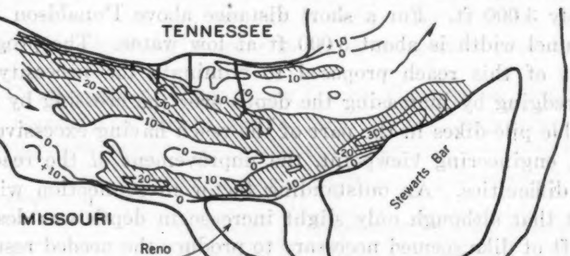

FIG. 15.—CONDITION OF STEWARTS BAR AS SHOWN BY SURVEY OF NOVEMBER, 1933.

was in such excellent condition as to cause no concern. The dikes were noted to have been very effective in inducing a heavy fill, and had maintained that crossing at a depth of 10 ft or more all that season, despite extreme low-water conditions.

The Walkers Bar Model.—The project for the canalization of the Ohio River, from Pittsburgh, Pa., to Cairo, Ill., was completed in 1929. This project was not designed to provide the required depth of 9 ft at all bars, because it was judged more economical to secure the needed depth at the worst crossings by dredging rather than by increasing the height or number of dams. At one location, Walkers Bar, where excessive dredging had been required annually, it was thought that impermeable dikes might provide the required depths more economically.

At the request of the District Engineer of the Louisville, Ky., District, a model study was undertaken to ascertain the feasibility of the plan, and the most effective system of dikes for the improvement of this vicinity. An engineer from the Louisville District familiar with local conditions was detailed to assist in the experiment. The experiment extended over a 17-

month period and was completed in September, 1932. During this time twenty-two separate plans were studied exhaustively and at least two were found which, in the model, gave the desired results.

On May 31, 1934, the U. S. District Engineer Office, at Louisville, reported on an up-to-date survey of Walkers Bar to note the effect of erecting dikes, and the extent of the scour and fill that had occurred since the dikes were put in operation. Benefits to the channel were unquestioned although the survey was made before the end of the high-water season.

The Island No. 9 Model.—For many years deficient depths during the low-water seasons at the Donaldson Point Crossing (Mile 56 below Cairo, Ill.) have proved a hindrance to navigation. In the past it has been the custom to provide relief during low stages by channel dredging. Since the low-water season of 1927, a total of 1 750 000 cu yd of material has been dredged at this crossing. The crossing is situated near the lower end of a reach which, for about 10 miles, lies practically in a straight line. Throughout the greater part of the reach the average width between banks at low stage is approximately 3 000 ft. For a short distance above Donaldson Point Crossing the channel width is about 5 000 ft at low water. The program for the improvement of this reach proposed to eliminate the necessity for annual low-water dredging by increasing the depth over the crossing by constructing four permeable pile-dikes in the part of the reach having excessive widths.

From an engineering viewpoint, the improvement of the reach presented no unusual difficulties. An outstanding factor in connection with the plans was the fact that although only slight increase in depth was desired, a total of 8 500 lin ft of dike seemed necessary to produce the needed results. Hence, the problem was largely one of engineering economics.

After the problem was submitted to the Laboratory, the experiment was conducted with the objective of finding a more economical method of producing the same effect as would be obtained by the proposed 8 500 ft of dike. Several modified plans were found, all of which indicated that the required deepening could be attained at a considerably lower cost than was originally contemplated. The most promising of the several plans indicated that four short pile-dikes on the opposite side of the stream, having combined lengths of only 1 600 ft, built in conjunction with five sand and gravel dikes, would give the required deepening and, at the same time, would furnish protection to the bank in a certain locality where it was being actively caved.

This latter plan, modified somewhat as to physical appearance, but similar as to hydraulic effect, was adopted for field construction, and work was begun in the early part of 1933. A re-survey of the area was made during June, 1933. Although it was too early at this time, of course, to expect that the effects of the works had been accomplished, the survey showed that the structures were tending to produce the desired result although several seasons may may be required for its consummation.

Cut-Off Studies.—One of the most excellent opportunities for comparing effects produced by identical changes on similar models of differing scales is that afforded by the experiments designed to determine the results of

artificially made cut-offs on the Mississippi River. Among the first models built at the U. S. Waterways Experiment Station was a small replica of the Greenville Bends, its horizontal scale being 1:4 800 and its vertical scale 1:360. With this model, tests were made to determine the effects of all possible cut-offs in the Greenville Bends region.

Some months later a model study was ordered to determine effects of ten dredged cut-offs in the Mississippi River between Rosedale, Miss., and Point Breeze, La. These ten newly proposed cut-offs included two that had been tested by the first model. The second model was built outdoors to scales of 1:2 400 and 1:120, and constitutes, even at this time, the largest model of its kind in the world.

During 1933, two new models were built to a considerably larger scale than had been attempted previously for a study of this kind. These last models, designed to accommodate movable beds of sand, were constructed to a horizontal scale of 1:1 000, and a vertical scale of 1:100. The purpose of their operation was to determine the amount of bed lowering that might be expected up stream from the several cut-offs. Incidentally, water-surface lowerings were also observed in order to check the results previously obtained

TABLE 1.—COMPARISON OF EFFECTS ON GAUGE HEIGHTS OF CUT-OFFS IN THE GREENVILLE BENDS FOR A 1929 FLOOD DISCHARGE

River gauge	Cut-Off at Leland Neck		Cut-Offs at Leland and Tarpley Necks	
	1:4 800 model	1:2 400 model	1:4 800 model	1:2 400 model
96	0.0	0.0	0.0	+0.2
99	−2.5	−2.1	−3.2	−3.0
101	−2.2	−2.3	−3.6	−3.0
103	−2.2	−1.9	−2.9	−2.2
104	−1.8	−1.7	−2.1	−2.3

from the other models. Tables 1 and 2 show comparisons between results obtained from the several models.

TABLE 2.—COMPARISON OF EFFECTS OF CUT-OFFS

River gauge	River mile	Stage Changes, in Feet	
		1:2 400 model	1:1 000 model
(a) Effect of a Flood Discharge in 1929, at Diamond and Yucatan Points			
80	574.9	−1.9	−1.85
73	617.9	−3.5	−3.2
(b) Effect of Cut-Offs at Giles Bend, Under the Conditions of a 40-Foot Stage, at Natchez, Miss.			
67	657.2	−2.5	−2.5
St. Joseph	662.4	−2.6	−2.5
64	681.4	−3.9	−4.0
62	687.8	−4.0	−4.0

Following tests on the 1:2 400 model, certain cut-offs were actually made on the Mississippi River. Subsequent observations have been in line with what would be expected for the current state of development.

HYDRAULIC FLUME TESTS CORROBORATED BY OBSERVATIONS ON RHINE RIVER

Not all cases of similarity between model and prototype are as apparent as the foregoing, but, on the other hand, numerous instances are on record in which experimental data have been checked daily by observed phenomena in Nature. An interesting case of this kind occurred in connection with studies of Island No. 35, up stream from Memphis, Tenn. (see Fig. 16). The U. S. Waterways Experiment Station has a tilting hydraulic flume as a part of its fixed equipment. This flume was used in an extensive series of tests which had as its objective the study of the tractive force of flowing water and the transportation of bed-load material in natural streams. In one phase of the experiments the procedure was as follows: The bottom of the flume was covered to a given depth with sand which had been subjected previously to a mechanical analysis. The inclination of the flume was set to a designated slope, which remained fixed until the completion of the run. Water was then passed through the flume and regulated by means of the weir and the tail-gate so as to have the exact slope of the sand bed. The depth of the water was next increased by successive increments, observations being made and recorded of all essential hydraulic data for each observed depth. Fig. 17 shows in graphic form the results of a typical run. In this diagram depths are plotted as abscissas, and mean velocities, rates of bed-load movement, and experimentally derived values of Manning's *n*, as ordinates.

FIG. 16.—MODEL STUDY OF NAVIGATION DIFFICULTIES ENCOUNTED AT ISLAND NO. 35, MISSISSIPPI RIVER.

Several interesting facts were brought to light by this experiment. It was noted that the mean velocity curve broke first at the point where the type of flow changed from laminar to turbulent; the next break in this line occurred simultaneously with the beginning of movement of the bed material. Most authorities are in agreement that when movement of the bed material in a stream begins there is an accompanying reduction in the rate of increase of the mean velocity of the water, with a corresponding change in the roughness coefficient of the stream. This was plainly shown to have occurred in all the foregoing experiments. Although the graphs shown in Fig. 17 are the results of only a single test, several like experiments, in which different sands and different slopes were used, were also performed, and the results were similar in each instance. Fig. 18 is a graph presented by Ph. Forch-

heimer,[8] which shows data taken on the Rhine River at Basle, Switzerland. The data have been plotted in such a manner that the slope of the curve is a measure of the roughness. The break occurs at the time movement of the river bed begins. In commenting on this graph, Forchheimer states[8] in effect that Kutter knew that the coefficient, *C*, in the Chezy formula decreased when the "geschiebe" (bed-load material), began moving; du Boys, also, was familiar with this fact and mentioned it in his writing; and it is common knowledge among hydraulic engineers that the roughness of a river bed changes as the stage increases.

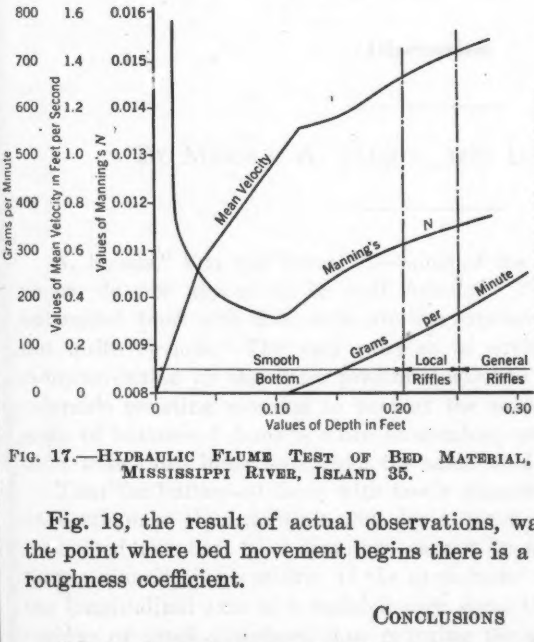

FIG. 17.—HYDRAULIC FLUME TEST OF BED MATERIAL, MISSISSIPPI RIVER, ISLAND 35.

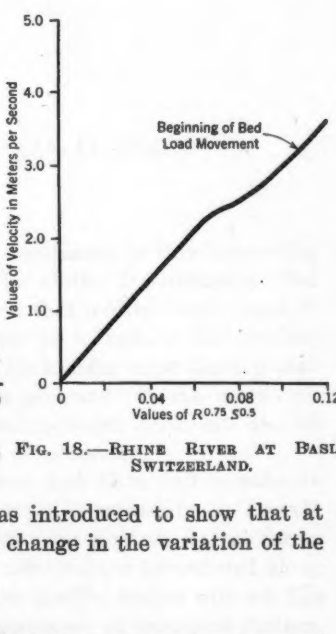

FIG. 18.—RHINE RIVER AT BASLE, SWITZERLAND.

Fig. 18, the result of actual observations, was introduced to show that at the point where bed movement begins there is a change in the variation of the roughness coefficient.

CONCLUSIONS

In the foregoing no attempt has been made to present an inventory of field verifications pertinent to model investigations conducted at the U. S. Waterways Experiment Station. That the list is incomplete is obvious and, in many cases not cited, sufficient similarity has been observed in Nature to compel faith in results of the tests. In rare instances verifications have been more difficult—if not altogether too difficult—to obtain. The main point to note is that the science, which was a veritable "infant" only a few years ago is now (1935) a lusty "youngster", rapidly coming of age, and demanding that its importance be recognized by every one.

[8] "Hydraulik", von Ph. Forchheimer, Leipzig und Berlin, 1930.

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DISCUSSIONS

WATER-BEARING MEMBERS OF ARTICULATED BUTTRESS DAMS

Discussion

By Messrs. A. Floris, and Hakan D. Birke

A. Floris,[11] Esq. (by letter)[11a].—Some of the statements in this interesting paper, do not appear to be well founded. The claim, for instance, that buttressed dams with deck slabs are less expensive that multiple-arch dams, is not quite obvious. The very purpose of arches is to reduce the bending moment caused by the water pressure; whereas deck slabs must develop considerable resisting moment to support the same pressure. If the mass concrete of buttressed dams is more economical, multiple-arch dams can also be built with thick buttresses using the same kind of concrete.

That the buttressed dams with freely supported deck slabs will be safer in an earthquake than multiple-arch dams is not at all a proved fact. Loosely connected members of a structure cannot be stronger against lateral forces than a monolithic structure. If the earthquake movement is propagated along the longitudinal axis of a multiple-arch dam, the flexible arches will act like springs or shock absorbers, thus relieving the structure of excessive stresses. The same action cannot be claimed in the case of buttressed dams.

In his analysis Mr. Birke determines the minimum cost of the deck slabs supported by buttresses with large heads. He considers these heads as cantilevers and applies the beam theory of prismatic bars. It cannot be taken for granted that the stress distribution in these large buttress heads will be the same as in a simple cantilever beam. Therefore, the conclusion that buttressed types of dams (including round-head buttresses), can be subjected to an exact mathematical analysis, may be seriously questioned. This being the case, the painstaking cost analysis of buttressed dams given in this paper appears to be of restricted usefulness to engineers engaged in the design of dams.

Note.—This paper by Hakan D. Birke, Jun. Am. Soc. C. E. (now Assoc. M. Am. Soc. C. E), was published in September, 1933, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1934, by R. A. Sutherland, P. Wilhelm Werner, Charles P. Williams, and Paul Baumann.

[11] Los Angeles, Calif.

[11a] Received by the Secretary February 23, 1934.

HAKAN D. BIRKE,[12] ASSOC. M. AM. SOC. C. E. (by letter).[12a]—An excellent group of discussions has been presented in connection with this paper. It is gratifying that the importance of establishing the principles of economic design of articulated buttress dams has been recognized by such competent authorities.

Aside from its merits in the field of economy and its high factor of safety, the principal advantage of the articulated buttress dam, as pointed out by Mr. Sutherland, is its adaptability to a wide variety of foundation conditions. The increasing scarcity of sites with sound rock foundations will undoubtedly lead to a more widespread use of this type of dam and, therefore, a detailed knowledge of its economic limits becomes of major importance.

In attempting to establish such economic limits Mr. Sutherland has compared a round-head buttress type dam, 120 ft. high, having a buttress spacing of 40 ft, center to center, with a gravity type dam of the same height. The buttress dam is found to be somewhat more economical than the gravity dam although Mr. Sutherland found the reverse to be true for all heights less than 90 ft. His analysis gave rise to the inquiry as to whether a reduction of buttress spacing at the flanks of the dam would be more economical than the use of the same buttress spacing throughout.

In answer to this inquiry it may be stated that there are several definite objections to changing the buttress spacing. In the first place, a change of spacing throws a heavy eccentric loading on the buttress at the point of transition. Such a transition is necessarily a heavy structure and the additional concrete required goes a long way toward offsetting the small economy that would be gained by changing the buttress spacing. A second and perhaps more serious objection is that a change in spacing would increase the cost of form work considerably. One of the advantages in the buttress type of construction is that standard forms may be used throughout the entire dam. This leads to an appreciable economy in construction costs.

In determining the most economic type and buttress spacing for a dam at a given site, the proper procedure is first to establish the mean height of the dam. Comparisons may be made of the mean heights for the various types; but it will be found, in general, that these heights will agree closely and will not be far enough apart to affect the selection of type. The mean height may then be used to determine the most economical height and the correct buttress spacing.

If Mr. Sutherland had investigated other types and buttress spacings, he would have found that a considerably closer spacing than 40 ft and an Ambursen type of deck and haunch would have shown a much greater saving than the round-head dam or the gravity dam used in his example for heights even as low as 25 ft.

Mr. Werner makes an analysis of the economical stress relations in concrete and steel (f_c and f_s) for a deck slab in flexure. By his analysis he demonstrates that in the example introduced by the writer (in which,

[12] Design Engr., Ambursen Dam Co., New York, N. Y.

[12a] Received by the Secretary December 5, 1934.

$f_s = 16\,000$; $f_c = 650$ lb per sq in.; and $\beta = 24.6$), better economy would have been obtained by using $\beta = 20$; but this value of β will yield a concrete stress, that is in contradiction to the assumption. Mr. Werner's analysis is based on a constant value of f_s, and, therefore, using his suggested value of β would make $f_c = \dfrac{16\,000}{20} = 800$ lb per sq in., which is in excess of the assumed maximum allowable concrete stress, $f_c = 650$ lb per sq in.

Generally, in design problems, maximum allowable stresses are used regardless of their economic relations. Mr. Werner has shown how the greatest economy for a deck slab in flexure may be attained when one of the stresses is kept constant and the other, varied. It is of equal interest to determine the economic effect of varying the stresses independently of each other. Consider, for example, Equation (69), as developed by Mr. Werner, and let,

$$Y = \frac{1}{\sqrt{z\,(3 - z)}} \left(1 + H \, \frac{z^2}{1 - z}\right) \frac{1}{\sqrt{f_c}} \qquad \text{.............(117)}$$

or, by introducing:

$$z = \frac{n}{\beta + n} \qquad \text{........................(118)}$$

the result will be:

$$Y = \frac{\beta\,(\beta + n) + H\,n^2}{\beta \sqrt{n\,(3\,\beta + 2\,n)\,f_c}} \qquad \text{................(119)}$$

and,

$$C_{sd} = Y \sqrt{\frac{6\,M}{l}}\; l\, L\, D_c \qquad \text{....................(120)}$$

For a certain ratio, $\dfrac{D_s}{D_c}$, the cost of the slab may be determined from Equation (119) for various stress values.

Assuming that $C = 0.2$, $n = 15$, and $\dfrac{D_s}{D_c} = 66$, Equation (119) has been evaluated for $f_s = 10\,000$ to $20\,000$ lb per sq in., and $f_c = 400$ to 800 lb per sq in. The results have been plotted and then curves of equal costs have been drawn, as shown by Fig. 13.

The stress condition introduced by the writer in his typical example (that is, $f_s = 16\,000$ and $f_c = 650$ lb per sq in.), is represented by Point A in Fig. 13. It may be seen that reducing any or both of the stresses would only increase the cost of the slab. Any increase in the stresses is out of the question as these values were assumed to be the highest allowable. Thus, it is proved that the stress condition used by the writer was the most economical one.

Fig. 13 demonstrates some interesting facts about the economy of different stress relations for a slab in flexure. Thus, for a low value of the stress in concrete, such as, $f_c = 450$ lb per sq in., little economy would be gained

by stressing the steel above 14 000 lb per sq in. In the same way, for a value of f_s = 13 000 lb per sq in., no economy will be gained by stressing the concrete above 600 lb per sq in.

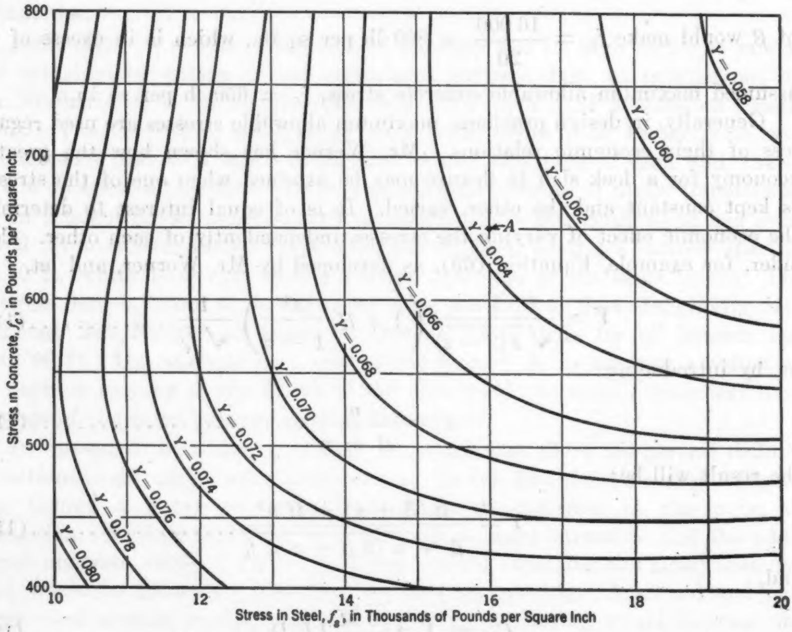

FIG. 13.—CURVES OF EQUAL COST FOR DECK SLAB IN FLEXURE AT VARIOUS VALUES OF STRESSES IN STEEL AND CONCRETE RATIO, $\frac{D_s}{D_c} = 66$.

The object of the paper was to demonstrate the conditions under which maximum economy would be attained for the water-bearing members of articulated buttress dams. The economic equations were based on ordinary designing assumptions which, in some cases, are somewhat approximate. This has been pointed out by some of the discussers, and, therefore, it is necessary to consider the designing methods presented by the writer in the light of these criticisms.

Mr. Williams has pointed out that the equations developed in the paper are based on the conventional formulas for designing beams of uniform depth, and then demonstrates the differences between these and the formula for wedge-shaped beams as developed by the late William Cain, M. Am. Soc. C. E.

If the haunch is designed as a wedged beam by the methods introduced by Professor Cain, the shear stress on the tensile side of the neutral axis is determined by Equation (81), as presented by Mr. Williams. This shear stress, τ_1, generally will control the depth of the section because the shear stresses on the compressive side of the neutral axis are combined with compressive normal stresses and hence will not cause any diagonal tension.

The total shear, S, at Section *A-A*, Fig. 12, will be equal to $\frac{1}{2}\ p\ o\ a$ and the moment, M, will be equal to $\frac{1}{3}\ p\ o\ a^2$. If these values are introduced in Equation (81), and d is eliminated by combining Equations (81) and (83):

$$\tau_1 = \frac{\cos\ \phi}{2}\sqrt{\frac{3\ k\ f_c\ p_o}{2\ j}} - \frac{k\ f_c \sin \phi \cos \phi}{2} \quad \text{..........(121)}$$

If the haunch is designed as a beam of uniform depth the shear, τ, will have the value:

$$\tau = \frac{S}{j_1\ d} = \frac{p_o\ a}{2\ j_1\ d} \quad \text{....................(122)}$$

or, by combining with Equation (83),

$$\tau = \frac{\cos\ \phi}{2\ j_1}\sqrt{\frac{3\ k\ j\ p_o\ f_c}{2}} \quad \text{..................(123)}$$

From Equations (80), (121), and (123):

$$\frac{\tau_1}{\tau} = \frac{j_1}{j} - j_1\sqrt{\frac{6\ f_s\ (1 - j)^2}{p_o\ n\ j\ (3\ j - 2)}}\ \tan \phi \quad \text{..........(124)}$$

In Equation (124), τ_1, designates the shear when the haunch is designed as a wedged beam and, τ, the shear in the haunch computed as for a beam with uniform depth. Thus, Equation (124) gives the relation between the controlling shear stresses under these different assumptions.

In the economy problem, the factors, f_s, p_o, n, and ϕ are constants and thus the only variables in the expression for $\frac{\tau_1}{\tau}$ (Equation 124) will be j_1 and j, which are the distribution factors for shear. As the variations in j_1 and j will be negligible, the relation will be nearly constant. If the haunch is to be designed for a certain value of τ_1 (controlling shear when computed as a wedged beam), from Equation (124) the corresponding value of τ (controlling shear when the haunch is designed as a beam of uniform depth) may be determined. Using this value of shear, the depth of the haunch may be determined as if it were a beam of uniform depth, and the formulas developed will still apply.

Mr. Williams demonstrates, that in determining the depth of the haunch only two of the three factors j, τ, and f_s can be assumed. However, that should not restrict the use of Equation (30) for determining the steel area in the haunch for bending moment. For instance, if the expression, $S = \frac{p_o\ a}{2}$, is introduced in Equation (97), as presented by Mr. Williams, the result will be:

$$\frac{(1 - j)^2}{3\ j - 2} = \frac{8}{27}\frac{n\ \tau^2}{f_s\ p_o} \quad \text{........................(125)}$$

in which, the maximum pressure p_o, on the haunch, is constant. If the values of τ and f_s are assumed, it is a simple matter to solve Equation (125) and determine the value of j. There is then no objection to using Equation (30).

The load the deck must carry, in pounds per square inch, may be written as follows (see Fig. 4(a), and Fig. 14):

Water Load:

$$62.5\ (H - t \cos \alpha) \qquad (126)$$

Weight of Structure:

$$150\ t \cos \alpha \qquad (127)$$

Total load:

$$W = 62.5\ H + 87.5\ t \cos \alpha \qquad (128)$$

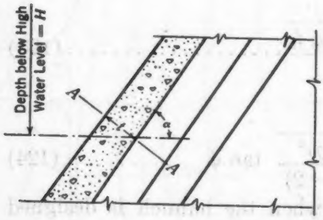

FIG. 14.—VERTICAL SECTION THROUGH DECK OF AMBURSEN DAM.

If, as an approximation, the total load is assumed as, $W = 62.5\ H$, it does not mean, as stated by Mr. Williams, that the water-load alone has been taken into account (compare Equation (126)). It only means that the last term in Equation (128) has been neglected. For the Ambursen dam introduced as an example by Mr. Williams, having a buttress spacing of 22 ft, on centers, the last term in Equation (128) varies from the value 0.06 W, at 30 ft below the top, to 0.03 W at a depth of 180 ft. At greater heights, where the cost of the water-bearing members is a more important item, the last term will be entirely negligible when compared to the first.

Mr. Baumann has demonstrated the difficulties of determining the exact bearing-pressure distribution on the haunch. The writer assumed a triangular bearing pressure area according to the common practice for this kind of structure. Actually, the character of this pressure distribution will vary with the relative rigidities of the haunch and deck. The writer is of the opinion, however, that an assumption of triangular distribution is on the safe side.

Mr. Baumann has made a check on the result of this assumption by comparing the deflection of the deck with the yield of the haunch at corresponding points. It may be noted that the formulas for these deflections are based on an assumed straight linear distribution of the stresses. Actually, in the deck above the bearing area of the haunch, the distribution of the stresses is not nearly as simple as straight-lined diagrams would indicate. In view of this, the results of Mr. Baumann's computations do not offer sufficient evidence to justify changing the assumption of triangular bearing pressure on the haunches.

It is quite true, as Mr. Floris states, "that the buttressed dams with freely supported deck slabs will be safer in an earthquake than multiple-arch dams. is not at all a proved fact." Neither is the converse a proved fact. Under some conditions, loosely connected members of a structure will not be as strong against lateral forces as a monolithic structure, but it is not a fact that loosely connected members cannot be stronger, against all forces acting, than a monolithic structure. To be stable a monolithic structure must have

sufficient strength, in all its parts, to resist all the loads to which it will be subjected. In many cases, construction joints, properly located, render a structure safe, whereas, if the joints were not provided, the structure would be unsafe. This is a well recognized and generally accepted fact in structural design.

Mr. Floris states that: "If the earthquake movement is propagated along the longitudinal axis of a multiple-arch dam, the flexible arches will act like springs or shock absorbers, thus relieving the structure of excessive stresses". This description might be correct, possibly, if the earth tremors were slight; however, the arches could scarcely withstand a severe shock without being overstressed, and it is improbable that they could withstand the torsional stresses that frequently accompany earthquakes. Even assuming that the only earth movement is in the direction of the longitudinal axis, the members connecting the buttresses, in transmitting the shock, would act as columns. It is inconceivable that a curved column composed of arch barrels would transmit a shock more safely than a straight column composed of flat-slab decks.

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DISCUSSIONS

AN APPROACH TO DETERMINATE STREAM FLOW

Discussion

BY MESSRS. GORDON R. WILLIAMS, AND W. G. HOYT, L. L. HARROLD, AND F. F. SNYDER

GORDON R. WILLIAMS,[31] JUN. AM. SOC. C. E. (by letter).[31a]—The unit-graph method has interesting possibilities in the determination of flood flows. The author considers the feasibility of establishing magnitude-frequency relations through the use of an equation for rainfall intensities and the pluvia-graph, thereby assuming a storm coefficient of unity. In regions where melting snow is a component of stream flow the assumption of a storm coefficient of unity or greater than unity is reasonable, but there are many places where such an assumption would not be in accord with hydrologic experience. In such places it might be possible, by the author's methods, to determine the frequency with which storm coefficients approach maximum values. The next question is to determine the frequency of flood-producing storms. This necessitates a consideration of the intensities of rainfall, the areas covered by rains of various intensities, the positions of the storm centers, and the directions of storm movement. If the task of combining these factors so that they can be properly expressed in relation to frequency could be accomplished, there still would remain the problem of combining this result with the frequency of maximum storm coefficients in order to attain the ultimate goal, which is the frequency of flood discharges of various magnitudes.

The unit-graph method appears to be applicable to the determination of rare floods that can be expected from moderate-sized drainage areas. The author states that the method has been applied for other purposes to drainage areas approximating 6 000 sq miles. The determination of the probable maxi-

NOTE.—The paper by Merrill M. Bernard, M. Am. Soc. C. E., was published in January, 1934, *Proceedings*. Discussion in this paper has appeared in *Proceedings*, as follows: March, 1934, by C. S. Jarvis, M. Am. Soc. C. E.; April, 1934, by LeRoy K. Sherman, M. Am. Soc. C. E; May, 1934, by W. W. Horner, M. Am. Soc. C. E.; September, 1934, by Messrs. C. H. Eiffert and Charles S. Bennett; and November, 1934, by R. L. Gregory and C. E. Arnold Assoc. M. Am. Soc. C. E. The discussions in this (January) issue are published by permission of the Director, U. S. Geological Survey.

[31] Asst. Engr., U. S. Geological Survey, Washington, D. C.

[31a] Received by the Secretary December 20, 1934.

mum flood is a preliminary step in deciding upon suitable designs for most hydraulic structures. It would be of value in almost any method for analyzing floods, even in purely theoretical procedures, such as that recently presented by Professor J. J. Slade, Jr.[32]

Flood studies bring out the fact that the smaller the drainage area the greater the potentialities for producing high yields per unit area. Small drainage areas usually have the important requisites for producing floods; that is, they can be entirely surrounded by areas having higher rates of rainfall, they have steep slopes, and they lack the essentials for natural flood detention. Many of the most severe floods are those produced by so-called "cloudburst" storms. The periodicity of such floods may be very great when expressed in years, so great, in fact, that a study by statistical methods[33] of what might be considered a long record in the United States would not indicate that such a flood is likely to occur. Examples of these rare floods occurred on the Miami River, in Ohio, in 1913,[34] on the Arkansas River, at Pueblo, in 1921,[35] and on many rivers in Vermont in 1927.[36] The discharges in these floods were two or three times greater than would be expected from a study of a magnitude-frequency relation. For such floods, statistical methods appear to be inadequate. The unit-graph method, on the other hand, may make it possible to arrive at an approximation to the limiting flood through the application of high rainfall rates, together with high, but altogether possible, storm coefficients, and the placing of a hypothetical storm in a critical position.

On large drainage areas, such as that of the Ohio River or the Tennessee River, where the application of the unit-graph method might encounter difficulties, the statistical method seems to offer a partial solution, provided a long record is available. On such drainage areas the differences in percentage between the leading floods in the record are usually small in comparison to those from a small drainage area, thus making the periodicity a relatively less important factor in relation to magnitude. A frequency curve derived from perhaps a 40-yr to 50-yr record of a large drainage area might be extended to a periodicity of materially greater length, with the assurance that an approximation to the maximum flood had been reached.

W. G. Hoyt,[37] M. Am. Soc. C. E., and L. L. Harrold,[38] and F. F. Snyder,[39] Juniors, Am. Soc. C. E. (by letter).[39a]—In connection with general studies of

[32] *Proceedings*, Am. Soc. C. E., October, 1934, p. 1118.

[33] See, "Flood Flows", by the late Allen Hazen, M. Am. Soc. C. E., John Wiley & Sons, Inc., 1930; "Theoretical Frequency Curves and Their Application to Engineering Problems", by H. Alden Foster, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 87 (1924) p. 142; and "Flow in California Streams", *Bulletin No. 5* (1923), California Dept. of Public Works, Div. of Eng. and Irrig.

[34] "The Ohio Valley Flood of March-April, 1913", by A. H. Horton, Assoc. M. Am. Soc. C. E., and H. J. Jackson, U. S. Geological Survey, *Water Supply Paper 334* (1913).

[35] "The Arkansas River Flood of June 3–5, 1921", by Robert Follansbee, M. Am. Soc. C. E., and Edward E. Jones, U. S. Geological Survey, *Water Supply Paper 487* (1922).

[36] "The New England Flood of November, 1927", by H. B. Kinnison, Assoc. M. Am. Soc. C. E., U. S. Geological Survey, *Water Supply Paper 636 C*.

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[38] Asst. Engr., U. S. Geological Survey, Washington, D. C.

[39] Junior Engr., U. S. Geological Survey, Washington, D. C.

[39a] Received by the Secretary December 31, 1934.

relations between rainfall and run-off being made by the U. S. Geological Survey, in collaboration with the Water Planning Committee of the National Resources Board, the writers have had an opportunity to investigate the distribution graph devised by Mr. Bernard based on the unit graph set forth by Leroy K. Sherman,[3] M. Am. Soc. C. E., to study relations between precipitation and surface run-off. Although the studies have not progressed far enough to warrant definite conclusions, it is believed that a brief summary of some of the problems encountered will be of value to hydrologists and engineers who may be interested in water utilization. It should be recognized that the distribution graph described by Mr. Bernard effects the same purpose as Mr. Sherman's unit graph, and that, in theory at least, both relate wholly to direct surface run-off, because in the development of a unit graph all flow occasioned by precipitation previous to the storm under consideration is segregated from the flow resulting from the 1-day storm. The accuracy with which a unit graph and a distribution graph can be developed, therefore, depends upon the accuracy with which the stream flow resulting from antecedent rainfall can be determined—a problem with which many hydrologists and hydraulic engineers have been concerned.

By definition and example, Mr. Bernard develops a so-called "recession curve" by combining the recessions of the hydrograph of several storm periods followed by intervals of no rainfall, which he states "gives a dependable idea of how base flow would have fallen and the flood flow would have receded had no subsequent rainfall occurred," and he uses this curve to determine the so-called "base flow". It would seem that this method properly excludes surface run-off resulting from antecedent rainfall, but does not take into account the facts that base flow—or, as otherwise defined, ground-water flow, or seepage flow—is a variable quantity and that recession curves based on hydrographs of stream flow, unless made up of segments of the hydrograph representing periods when no surface run-off is in the stream, may not be comparable with depletion curves of ground-water flow. For example, in drawing the base-flow curve for the Muskingum River at Dresden, Ohio, Mr. Bernard assumes that the base flow would have fallen from 1 500 to 700 cu ft per sec in 11 days during March, 1929, and similarly during the two storms in July, 1925, and in July and August, 1927, that were used to determine the average-distribution graph. This use of a fixed "recession curve" does not take account of the fact that during March the water-table in the Muskingum Basin may be high enough to maintain a base flow between 2 000 and 5 000 cu ft per sec, whereas, in July, it may be only high enough to maintain a flow of 1 000 cu ft per sec, or less. Use of a fixed "recession curve" in the determination of the unit graph seems to rely on the accuracy of an average curve and to ignore departures therefrom due to variations in gound-water levels and ground-water flow and to recharge from the storm under consideration. If the storm producing the surface run-off occurs at a time when the ground-water flow corresponds to the lower end of the fixed recession curve, the error introduced may be negligible, but if the storm producing the surface run-off occurs when

[3] "Stream Flow from Rainfall by Unit-Graph Method," *Engineering News-Record*, April 7, 1932, p. 50.

the ground-water is high, the error may be very appreciable. In each case, of course, the error depends on the magnitude of the storm run-off in relation to the ground-water flow. Different methods[40] have been used to segregate surface run-off from ground-water run-off, but apparently none of them is subject to exact analysis, and care must be taken that the method used eliminates, and does not include, any ground-water flow in the determination of the unit graph.

If the fundamental requirements are observed the writers have found, as Mr. Bernard did, that the theory of the unit graph or distribution graph seems to apply to a remarkable degree in actual practice. In their derivation, however, the following factors must receive due consideration:

1.—In deriving unit graphs it is necessary to select storms that conform as closely as possible to the ideal, which is rainfall of uniform intensity and 24-hr duration. Since such storms do not occur in Nature over any extended area, the peaks of unit graphs of 1-day storms vary somewhat, depending on whether the storm center is near or remote from the gauging station. Conversely, in the application of an average unit graph to actual precipitation, the results will differ from the recorded data to the extent that the storm pattern differs from the average condition used in determining the unit graph.

2.—The published records of the U. S. Weather Bureau do not indicate whether the recorded precipitation on any one day occurred during 1 hr, or was well distributed over the 24 hr. In a similar manner an ideal 24-hr storm may be recorded in equal amounts on two consecutive days. Unit graphs for storms lasting 1 hr may not be comparable with those for storms lasting 24 hr, and in the application of the determined unit graph the resultant pluviagraph will differ from the recorded flow to the extent that the unit graph of storms of less than 24-hr duration differs from that of storms of 24-hr duration. Published records of precipitation give the total for the day, and published discharge records give the average flow for the day. These limitations in the details of recording basic data make it impossible to determine comparable unit graphs from small basins, but in the opinion of the writers they do not disprove the essential theory of the unit graph. These limitations also make it difficult to combine or average the separate unit graphs for the same basin. For example, a unit graph of one storm may show no rise on the day the precipitation was recorded, while that of another storm may show an appreciable rise, a difference which may result from inadequacies in the refinement of recording, or from causes not now definitely determined, such as varying initial rates of infiltration and interception, or water required and held in initial surface wetting.

3.—Theoretically, it would seem that, as a result of differences in channel velocities, the unit graph of a 1-day storm occurring when the river is at

[40] R. E. Horton, M. Am. Soc. C. E., in his discussions of "The Rôle of Infiltration in the Hydrologic Cycle", pub. in the *Transactions* of the Am. Geophysical Union, National Research Council, of the National Academy of Science, June, 1933, p. 446, outlines methods that have been used to separate surface flow from ground-water flow and gives valuable references.

a high stage would differ from that of a storm occurring when the river is at a low stage.

4.—Unit graphs of 1-day storms apparently cannot be obtained in basins where the infiltration capacity exceeds the rainfall rate, or where the direct surface run-off is substantially delayed either by artificial or by natural storage. To a moderate extent both these conditions occur in many basins, and the Rock River above Afton, Wis., is herewith discussed as an example of the difficulties which may be encountered in the preparation of distribution graphs of surface run-off resulting from 1-day storms. The drainage area is 3 190 sq miles. Lakes and marshes occupy between 10 and 15% of the area; channel slopes are small, and the soil is largely sandy loam. These conditions give rise to a large ground-water and sustained flow and a relatively small quantity of direct surface run-off. An examination of the daily hydrograph of the Rock River, at Afton, for the period, October, 1914, to September, 1933, failed to disclose an instance in which the rise in the hydrograph could be associated with a 1-day storm. All pronounced rises in the hydrograph were the result of rainy periods of from 4 to 10 days duration. Although in this particular instance it might be possible to develop distribution graphs based on 5-day storms, they would not seem applicable to any 1-day storm.

Notwithstanding these apparent limitations, distribution graphs having the same general pattern, have been obtained by the writers from basins ranging in area from 1 000 to 40 000 sq miles. Although the distribution graphs follow the same general pattern, the variations between peak magnitudes are very appreciable. These variations may be a source of inconsistencies in the application of an average distribution graph to hydraulic problems.

After developing an average unit graph or distribution graph, Mr. Bernard transposes the total weighted daily precipitation on the basin into so-called "pluviagraph values" and, in his examples, compares these values with total run-off by means of "retention coefficients" obtained by dividing the total observed run-off for the several days, making up the storm period by the total pluviagraph value or by distributed rainfall for the same period.

Two questions have puzzled the writers in regard to the pluviagraph: First, is it fundamentally logical to distribute total rainfall by means of a distribution graph based on direct surface run-off (rainfall minus infiltration and storage); and, second, is it logical to compare rainfall thus distributed with total recorded flow? In connection with the general studies previously mentioned, the writers found that the average annual direct surface run-off in all major sub-basins of the Mississippi River, above Keokuk, Iowa, ranged from about 2% of the precipitation for the Minnesota River Basin to about 26% of that for the Black River Basin. In other words, by the application of Mr. Bernard's method to the Minnesota River Basin, 98% of the pluviagraph would be built up of precipitation that did not appear as direct surface run-off. In the application to the Black River Basin, which apparently has the maximum surface run-off of all the larger basins in

the Upper Mississippi River drainage area, use of Mr. Bernard's procedure would distribute about 74% of the precipitation that did not appear as surface run-off.

Of the precipitation that did not appear as direct surface run-off, about 1% in the Minnesota River Basin and about 6% in the Black River Basin eventually appeared as sustained or ground-water flow. It seems illogical to the writers to include in a pluviagraph precipitation which, even if it does eventually appear as stream flow, arrives there through the ground and manifestly follows laws different from those controlling direct surface run-off as reflected in the unit graph.

For isolated storms, Mr. Bernard's "retention coefficient" is the simple relation between the total storm precipitation and the total run-off. For the Tuscarawas River, at Newcomerstown, Ohio, which he has used as an example, it is the relation between the total pluviagraph value and the total recorded run-off for the period between consecutive troughs on the hydrograph.

More and more hydrologists are looking upon stream flow as made up of two distinct parts, one of which results from direct surface run-off and the other from that part of the precipitation which is either retarded on the surface, or which passes through the ground and eventually reaches the stream channel. For the direct surface run-off the time between the occurrence of the precipitation and its appearance at the gauging station is relatively short, as shown by the unit graph. For the sustained or ground-water flow the time is much longer and can probably be measured in terms of months rather than of days. For example, the late J. F. Hayford,[a] M. Am. Soc. C. E., found, by strict mathematical analysis, that in the Wagonwheel Gap area, in Colorado, the flow on any one day depended, in part at least, upon the meteorologic conditions existing during the preceding 257 days. To the extent that the total flow is made up of ground-water flow dependent on conditions antecedent to the time covered by the distribution graph, relations between total pluvigraph records and total run-off seem valueless. If ground-water is deducted, however, comparisons between total pluviagraph records and observed stream flow less ground-water flow may have value. If they can be carried back through the distribution graph, these relations might show, to the extent that the surface run-off followed the average distribution graph, the amount of infiltration day by day. Furthermore, it may be possible to correlate the relations with past and present conditions to an extent sufficient at least to make use of total pluviagraph records in determining surface run-off from storms of great magnitude.

The writers are studying both methods of approach, and the results, to date (1935), seem to indicate, as Mr. Sherman and others have found, that antecedent conditions, as well as amount and intensity of the particular rainfall considered, play an important part. For example, under similar

[a] "A New Method of Estimating Stream Flow" by the late J. F. Hayford, M. Am. Soc. C. E., and J. A. Folse, Carnegie Inst., Washington, D. C., *Publication 400*.

storm conditions, low rates of infiltration with resulting high run-off relations, will apply on Ohio streams during normal spring periods, when soil moisture and ground-water levels are at a maximum, and high rates of infiltration with resulting low run-off relations will apply during normal Septembers, when conditions are reversed. However, if conditions are abnormal and moisture conditions in the spring are comparable to normal fall conditions, the infiltration and run-off relations for the two periods tend to be comparable. In so far as total surface run-off throughout the year is concerned, the applicability of the unit-graph method will depend upon the accuracy with which relations between several factors can be determined.

With respect to the value of the "retention coefficient" in connection with flood flows, a somewhat different condition exists. Floods result when infiltration is at a minimum, with corresponding high run-off relations. If there is no snow on the ground the pluviagraph record should represent, within reasonable limits, the maximum surface run-off possible with zero infiltration. It is believed that studies and experiments will eventually disclose infiltration limits that may be reached and thus make possible the determination of limiting run-off rates which, when used with Mr. Bernard's pluviagraph, will make it possible to determine flood run-off in a rational manner.

The author is to be commended for clarifying the basic principles of the unit graph, for his development of the distribution graph, and for his conception of the pluviagraph. Although it is far from an exact process for computing stream flow, the writers have found that it yields reasonably comparable distribution graphs, and that when net rainfall, as used by Mr. Sherman, or total precipitation, as used by Mr. Bernard, was distributed by means of an average unit graph or distribution graph, the peaks of computed and observed surface run-off occurred on the same day or, if not, rarely more than one day apart. The concepts developed are of material assistance in the analysis of run-off in relation to its causes, thus promoting a better understanding of the occurrence of trends and the possible affects of storage, land drainage, agriculture, and other factors that may be of importance in considering the wise utilization of water resources. They furnish a tool for analysis of rainfall and run-off that may be of practical value where detailed knowledge of hydrology is important, as in (*a*) manipulation of storage on large systems of river development for power and water supply; (*b*) obtaining definite knowledge of run-off characteristics of urban areas, where special conditions make such knowledge of exceptional value; and (*c*), analysis of potentialities of drainage basins for producing floods.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

WILLIOT EQUATIONS FOR STATICALLY INDETERMINATE STRUCTURES

IN COMBINATION WITH MOMENT EQUATIONS IN TERMS OF ANGULAR DISPLACEMENTS

Discussion

BY CHARLES A. ELLIS, M. AM. SOC. C. E.

CHARLES A. ELLIS,[5] M. AM. SOC. C. E. (by letter).[5a]—Thanks are due to those who have discussed this paper, for their kindly and constructive comments.

Mr. Eremin mentions that the equations are complicated and numerous. It is a fact that all mathematical expressions appear more or less complicated on first sight. The equations certainly are numerous; but the extraordinary structure rather than the method of analysis is responsible for the number.

Mr. Moisseiff has stated, in effect, that this new method provides the only possible path to a closely accurate determination of the behavior of a stiff frame under loads, especially when the frame is subjected to considerable deformation and departure from its original geometric shape. He has commented on the artificiality and incorrectness of solving for primary and secondary stresses separately. Mr. Tudor has also pointed out the great importance of "the fact that the method considers, simultaneously, primary (including participation) and secondary stresses." These points were not sufficiently emphasized in the paper and a further word may be clarifying.

The diagonal members below the floor (Fig. 3) were originally designed (see "First Approximation") on the commonly accepted assumption that they would resist the total horizontal wind shear in each panel. The degree to which this assumption proved to be in error (more than 100%), because of extraordinary stiffness in the vertical posts and their consequent capacity to resist shear, is indicated in the "Third Approximation" below Fig. 6.

NOTE.—The paper by Charles A. Ellis, M. Am. Soc. C. E., was published in January, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1934, by Messrs. Ralph A. Tudor, Leon S. Moisseiff, and A. A. Eremin.

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[5a] Received by the Secretary December 20, 1934.

The discovery and first application of the Williot equations came about, as follows: The transverse elastic curve was investigated as described in the first three paragraphs of the "Third Approximation." An ordinary Williot diagram for the lower portion was drawn after strains had been computed from approximate stresses and the quantities, ψ, were determined therefrom. Equations $\Sigma M = 0$, were written and solved; and the moment, M, at the ends of each member were computed. With M_{ko} and M_{ok} approximately known, it was possible to compute the shear, H_{ko}, in the member, KO, from Equation (5). It was obvious at once that this shear was a very considerable quantity, much too large to be neglected in assuming that the diagonals resisted the total horizontal wind shear. Repeated attempts by trial and error were made to estimate the shear distribution between the posts and diagonals. Each attempt included a revised set of stresses and strains, and a revised Williot diagram, etc. These attempts seemed to be getting nowhere, either because the writer's guesses went wide of the mark or because of the numerous factors in the problem.

After repeated drawings of the Williot diagram from revised guesses as to shear distribution, it became obvious that each ψ could be expressed algebraically in terms of the strains in the members, as shown in Equations (13) to (21); so, instead of determining the quantities, ψ, from a Williot diagram drawn from assumed strains, the question arose "Why not express each ψ in terms of unknown strains?" It was clear that the value of ψ_{oq} and ψ_{ko} had much greater influence on shear distribution than values of ψ for any or all other members. Hence, in the first application of the Williot equations, ψ_{oq} and ψ_{ko} were expressed as in Equations (17) and (18), and approximate numerical values derived from the preceding Williot diagram were assigned to the remaining ψ's. Equations (4) to (9) show how the expressions for stresses, P, and resulting strains, Δ, were developed. The result of this solution, since it contained Equation (4), determined the shear distribution, and revealed the glaring error originally made in assuming that the diagonals resisted all the external shear. It was then decided to "go the whole route" and use all equations—(13) to (21)—thereby eliminating all assumptions possible. This procedure was taken after major revisions had been made in the design of the diagonals and minor changes in the posts. The great value of the Williot equations was at once apparent, and they were used later to good effect in the analysis of the upper part of the structure.

Mr. Moisseiff's discussion of the labor involved in the solution of a structure with diagonal bracing compared with one having horizontal struts, should be qualified somewhat. If the two diagonals in any panel are removed and one horizontal strut is substituted, the number of members is reduced; and if the horizontal member is of the plate girder type and if no consideration is given to its depth, except its moment of inertia, then Mr. Moisseiff's statement, that a structure with diagonal bracing is more complicated and more laborious to compute, is correct.

On the other hand, when the horizontal member is a truss 30 ft deep having very large chord and web members, a thorough analysis becomes

much more complicated because, although two diagonals have been omitted in each panel, two post members have been added, together with a truss composed of numerous members. When the magnitude and novelty of the structure, and the consequent uncharted field of structural analysis with little or no precedent to follow, were considered, it was deemed wise to apply as thorough an analysis as possible. It was for these reasons that the Williot equations were applied also to the part above the floor.

Of course, the usual assumption that the diagonals resist the total shear in a panel would not result in such gross error in an ordinary truss; but it is pertinent to emphasize that any method (Manderla's included) of computing secondary stresses in a truss, which fails to take account of the influence of secondary stress upon primary stress, can no longer be considered as "most exact".

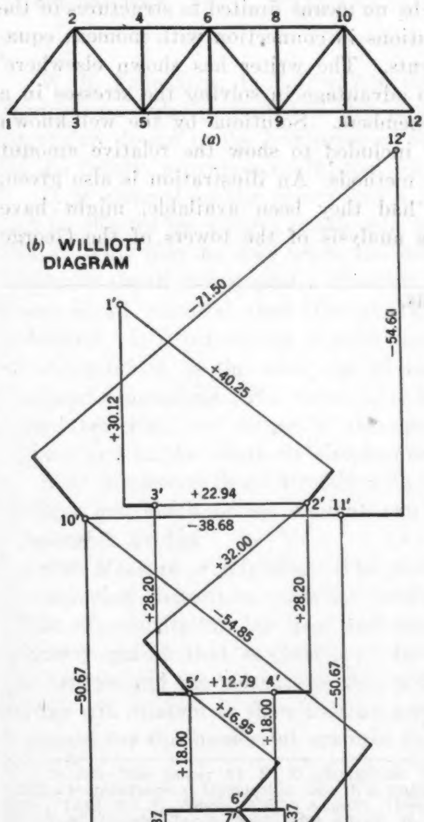

FIG. 16.—MOHR'S SEMI-GRAPHICAL METHOD OF ANALYZING SECONDARY STRESSES.

Mr. von Abo's valuable and scholarly paper, "Secondary Stresses in Bridges",[8] describes nearly all methods of secondary stress analysis known at that time, and gives solutions by the various methods, using the truss and load.[8a] The vertical shear in Panel 3–5 (Fig. 16(*a*)), for example, is 500 lb. The analysis shows that deformation imposes moments at the ends of the Members 2–4 and 3–5; consequently, these members are resisting transverse shear. These shears should be considered with the vertical component of the stress (primary and secondary) in Member 2–5, when balancing the total external shear of 500 lb in the panel. The true vertical component of the stress in this diagonal is not 500 lb, and any computation of secondary stress made on this assumption will be in error. The same conclusion applies to all members. The error here is small and of academic significance only, but Manderla's method can no longer stand as the model for accuracy. The analysis by the simultaneous solution is a closer approach to accuracy and will take on practical value as the chord

[8] *Transactions*, Am. Soc. C. E., Vol. 89 (1926). p. 69.

[8a] In Fig. 16(*a*) apply a load of 1 500 lb at Panel Point 9.

members become relatively stiffer and relieve the diagonals in transferring shear from one end of the panel to the other.

The primary and secondary stresses in the truss analyzed by Mr. von Abo may be solved simultaneously by using the Williot equations with Mohr's semi-graphical method as follows: In the Williot diagram (Fig. 16(*b*)) let the strains in the members be represented by Δ_{12}, Δ_{23}, etc. The quantities, ρ (and hence ψ), can be expressed, by the Williot equations, in terms of the Δ's and the thirty-three equations written. Twelve equations, $\Sigma M = 0$, can be written, one for each joint. They will contain twelve unknowns, ϕ, and twenty-one unknowns, Δ. The twenty-one additional equations may be derived from statics, since the transverse shear in any member may be expressed as a function of the end moments of that member. A solution of these equations will give the Δ's from which the stresses can be determined.

The use of the Williot equations is by no means limited to structures of the type presented in the paper, or to solutions in connection with moment equations in terms of angular displacements. The writer has shown elsewhere[7] how Williot equations may be used to advantage in solving the stresses in a simple truss having four redundant members. Solutions by the well-known methods of work and least work are included to show the relative amount of labor involved in each of the three methods. An illustration is also given, showing how the Williot equations, had they been available, might have been used to good effect in the stress analysis of the towers of the George Washington Bridge.

[7] *Engineering News-Record*, April 26, 1934.

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DISCUSSIONS

A GENERALIZED DEFLECTION THEORY FOR SUSPENSION BRIDGES

Discussion

BY MESSRS. W. R. FREDERICK, JR., AND WILLIAM H. YATES.

W. R. FREDERICK, JR.,[55] JUN. AM. SOC. C. E. (by letter).[55a]—The simplified working formulas presented in this paper are a great improvement over those heretofore found in the literature on the subject and, with the generalization of the theory, provide a useful tool for the engineer.

It is claimed in the paper that a saving of 5% in truss chord material can be effected by using a continuous truss (Design II) rather than a hinged truss. This may be true when the designs are developed in full detail and the truss depth is adjusted. However, Design I affords greater economy in truss chord material than Design II. This phase will be discussed subsequently. It is interesting to note that the 5% saving claimed would amount to about 0.25% of the total cost of constructing a suspension bridge of the assumed dimensions. The value of this paper lies not in the saving of truss chord material, but rather in the dependability of the results obtained by its use and in the relatively simple form in which it is presented.

This discussion deals directly with the three-span, symmetrical suspension bridge, but much of its content can be extended to include any type of suspension bridge.

The Measure of Rigidity.—The stiffening trusses are used for the purpose of reducing deflections, thereby providing a facility for, and inspiring a sense of security in, the user and insuring for him the least of maximum roadway grades that is practical. Determination of the desired stiffness of the trusses and the measure of this stiffness should be based on this fact. The writer will attempt to show that, in a structure of the approximate dimensions assumed for the numerical example in the paper, the vertical deflections are

NOTE.—The paper by D. B. Steinman, M. Am. Soc. C. E., was published in March, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: May, 1934, by E. Pavlo, Esq.; August, 1934, by Messrs. Jonathan Jones, A. Müllenhoff, H. Cecil Booth, Jacob Feld, and Glenn B. Woodruff, Howard C. Wood, and Ralph A. Tudor; September, 1934, by Messrs. L. J. Mensch, A. A. Eremin, Hans H. Bleich, F. H. Frankland, Gustav Lindenthal, Julian W. Shields, A. W. Fischer, and J. M. Frankland; November, 1934, by Messrs. Fredrik Vogt, Leon S. Moisseiff, and A. Mitchell and G. T. Parkin; and December, 1934, by Messrs. Sterling Johnston, Harold E. Wessman, and C. H. Gronquist.

[55] Park Engr., Long Island State Park Comm., Babylon, N. Y.

[55a] Received by the Secretary November 19, 1934.

practically meaningless to an individual using or viewing the bridge and that only the angular deflections are appreciable to his senses and thus affect his comfort and feeling of security.

Vertical Deflection.—The maximum vertical live load deflections are caused by full span loads, and the length of time required for them to occur under most favorable conditions is between 10 and 20 sec. An individual standing in the middle of a span would be dropped a distance of about 3 ft in this length of time. Consideration of elevator speeds and the nature of the acceleration will show this movement to be insensible. A car moving across the bridge with a large group of other vehicles placed so as to occasion maximum deflection at the center of a span, would traverse a continuous curve varying slightly from the roadway grade of the unloaded bridge. The variation of the vertical deflection, or of the curve from the grade, cannot be detected by any normal human sense.

The contribution of temperature to the vertical deflection need not be considered. It occurs slowly and depends less on the moment of inertia of the stiffening truss than the live load deflection. The vertical deflection does affect the structure in that, in order to maintain the minimum vertical clearance when fully deflected, it may be necessary to raise the theoretical profile. The expense involved is slight.

Angular Deflection.—Angular deflections, especially breaks in continuity, are visible and, if not unsightly, at least, may indicate weakness to the beholder. Furthermore, the angular deflection increases the grade for some vehicles using the structure. For others, the grade will be relieved. The conclusion is that vertical deflections are not satisfactory for measuring the rigidity and that angular deflections supply the most useful yardstick for this purpose. Designers may prefer to follow an intermediate course and use both vertical and angular deflections in measuring rigidity.

Deflections of Continuous and Hinged Stiffening Trusses.—Suspension bridge trusses should be designed of such stiffness that the angular deflection that increases the grade near the tower is under a specified maximum allowable deflection (or what amounts to the same thing), that the maximum grade along the deflected structure does not exceed a definite amount. This general rule should be modified by judgment and experience since extremely flexible trusses may result from assuming a high value for the allowable grade increase (Fig. 14). The maximum angular deflection at the approach end of the side span does not, for the usual structure, increase the roadway grade. However, the abrupt change of grade due to this deflection should be considered.

The points at which the grade increase results in the maximum grade for a vehicle to climb, are near the towers. The angular changes in the trusses may be greater at other points, but the resulting grades are not likely to be maximum since in most suspension bridges there is a camber of several feet in the main span and the side spans are on a grade up to the center span.

If the principle of limiting the maximum grade of the deflected structure is applied to the hinged truss, the controlling angular deflection is likely to

be at the tower end of the side span. If the side spans are shorter than one-half the main span this may not be true. Since the deflection curve for a hinged truss is broken at the towers while the curve for a hingeless truss is continuous, the hingeless truss has an immediate advantage in appearance if the angular change allowed is the same for both types of trusses.

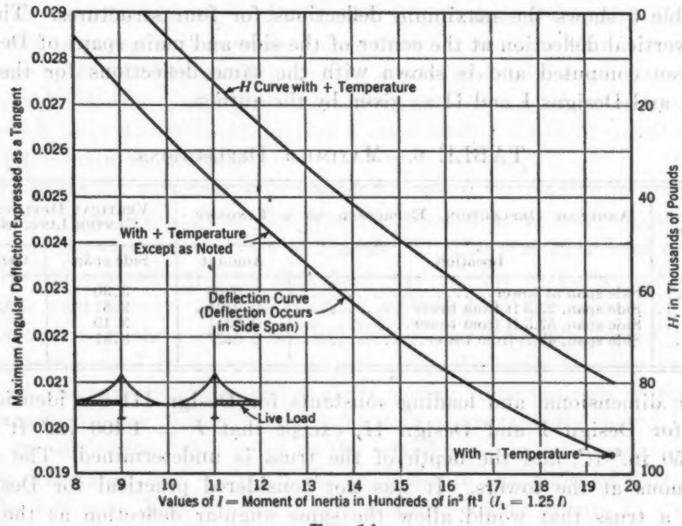

FIG. 14.

In attempting to reduce angular deflections to a practical minimum it will be well to keep in mind that the reduction of side span relative to center span will accomplish much by shifting the point where the maximum grade occurs away from the side span toward, and even on to, the main span. Fig. 15 showing the angular deflection plotted against the load position will

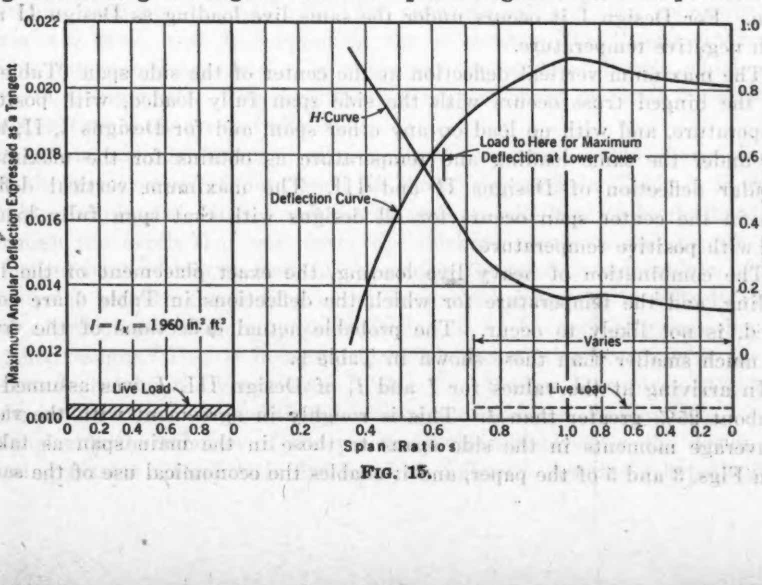

FIG. 15.

demonstrate the argument. The point noted on the curve as the slope change at the left tower is much less than the maximum slope change which occurs in the side span, 52.5 ft away from the tower under a different loading. Shorter side spans also straighten the side cables and thus have the secondary effect of stiffening the structure.

Table 6 shows the maximum deflections for four structures. The maximum vertical deflection at the center of the side and main spans of Design III has been computed and is shown with the same deflections for the hinged design and Designs I and II as given by the author.

TABLE 6.—MAXIMUM DEFLECTIONS

Design	Angular Deflection, Expressed as a Tangent		Vertical Deflection at Center Line, in Feet	
	Location	Amount	Side span	Center span
Hinged....	Side span at tower....................	0.0268	3.30	5.16
I........	Side span, 22.3 ft from tower..........	0.0194	2.81	5.28
II........	Side span, 52.5 ft from tower..........	0.0211	3.10	5.44
III........	Side span, 48 ft from tower............	0 0227	3.31	5.66

The dimensional and loading constants for Design III are identical with those for Design I and Design II, except that $I = 1\,400$ in.^{2} ft^2 and $I_1 = 1\,750$ in.^{2} ft^2, and the depth of the truss is undetermined. The truss is continuous at the towers. It was not considered practical for Design III to use a truss that would allow the same angular deflection as the hinged truss (Table 6). The resulting moments of inertia would have been $I = 950$ in.^{2} ft^2 and $I_1 = 1\,190$ in.^{2} ft^2 (Fig. 14).

The maximum angular deflection (Table 6) occurs for the hinged truss with one side span fully loaded, with positive temperature, and with no load on the other two spans, and for Design II and Design III with both side spans fully loaded, with positive temperature, and with no load in the main span. For Design I it occurs under the same live loading as Design II and with negative temperature.

The maximum vertical deflection at the center of the side span (Table 6) for the hinged truss occurs with the side span fully loaded, with positive temperature, and with no load on any other span, and for Designs I, II, and III under the same loading and temperature as obtains for the maximum angular deflection of Designs II and III. The maximum vertical deflection in the center span occurs for all designs with that span fully loaded and with positive temperature.

The combination of heavy live loading, the exact placement of the live loading, and the temperature for which the deflections in Table 6 are computed, is not likely to occur. The probable actual deflections of the truss are much smaller than those shown in Table 6.

In arriving at the values for I and I_1 of Design III, I_1 was assumed to be about 25% greater than I. This is roughly in agreement with the ratio of average moments in the side spans to those in the main span as taken from Figs. 3 and 5 of the paper, and it enables the economical use of the same

depth of truss for the side and for the center spans. In Design II the author has chosen $I = I_1$, and of such a value that the percentage deflection decrease at the center of the side span and the percentage deflection increase at the center of the main span relative to the same deflections of the hinged design are nearly equal. Granting that vertical deflections measure the rigidity, the result obtained in Design II is pointless unless the depth of truss in the main span is increased over that in the side span. A design of sections to obtain the condition, $I = I_1$, to maintain a constant truss depth throughout, and to carry the moments at stresses equal to, or less than, the allowable unit stress, would involve a considerable waste of metal in the chords of the main span truss.

Fig. 15 shows the variation with respect to load of the maximum numerical values of the negative angular deflection, expressed as a tangent, near the left tower of the continuous truss of Design II. The changing load consists of live loading extending from the right end of the structure. The left end of this load retreats from a position at 0.35 l of the main span until no load remains on the right. The full load on the left span does not vary. The temperature used in computing values along the curve is negative. The H-curve for the same loading conditions is shown. The position of the load for maximum angular change at the left tower is marked and noted. When the load extends beyond this point to the left the maximum angular change is in the main span and when all the load lies to the right of this point, except the left span load, the maximum angular change occurs in the side span. The greatest angular deflection occurs, however, when the temperature is positive and the value shown in Table 6 is this maximum.

Comparison of Designs.—Table 7 shows the relation between the moments of inertia, moments, truss depths, and chord material in the side and main spans of four designs. The data presented are intended to serve as a guide in determining relative economy. The moments shown are the averages of the maximum moments in each span for each design and the moments of inertia are those used in computing the moments. The theoretical truss sections determined are used to compute new moments of inertia which are shown in Column (7). The theoretical weight of chord material for a complete structure is shown in Column (9).

If carbon and high-strength alloy steels are used in a truss of any single span, the preparation of data such as those presented in Table 7 is considerably complicated, since a number of sections will have to be designed to determine the depth that will carry the imposed moments and supply the average moment of inertia used in computing the maximum moments.

A choice between the designs based on vertical deflection is difficult. The angular deflections (Table 6) show Design I to be definitely superior to the other designs. This is in agreement with the relative economy as far as this is affected by the chord area (Table 7). Design I requires 21% less weight in the chord sections than its nearest competitor, Design III.

To clarify the relation of the various factors involved in the design of stiffening trusses a formula is developed, as follows: Assume a hypothetical

TABLE 7.—RELATION BETWEEN MOMENTS OF INERTIA, MOMENTS, TRUSS DEPTHS, AND CHORD MATERIAL

(Unit stresses, in pounds per square inch; tension, 27 000 on net section; and compression, $27\,000 - 100\,\frac{l}{r}$, with a maximum of 23 000.)

Design (1)	Span (2)	Moment of inertia I, in inches² feet² (2)	Average of maximum moments, in foot-kips (4)	Depth, d, in feet (5)	Total theoretical area, top and bottom chords, in square inches (6)	Moment of inertia, I, for section designed, in inches² feet² (7)	Chord weight per foot of one truss, in pounds (8)	Total weight of chords of two trusses, three spans, in pounds (9)
Hinged	Side	2 420	10 475	10.4	89.5	2 420	304	702 000
	Main	1 960	6 375	14.0	39.8	1 944	135	
I	Side	2 420	8 850	12.7	60.1	2 420	204	523 000
	Main	1 960	6 083	14.7	36.1	1 945	122.5	
II	Side	1 960	8 763	10.1	76.7	1 957	261	692 000
	Main	1 960	7 315	12.5	50.3	1 962	171	
III	Side	1 750	8 203	9.7	74.8	1 759	254	661 000
	Main	1 400	5 942	10.9	46.9	1 387	159.5	

truss having a moment of inertia, I; depth, q; chord areas, A_t, each, and carrying a moment, M, at a chord stress of σ; all in appropriate units. Then, $I = \frac{1}{2}\,A_t\,q^2$ and $A_t = \frac{M}{\sigma\,q}$. If all factors are varied as indicated by subscripts, except σ, the following equation can be written showing the relation between the old and new chord areas:

$$A_{t1} = A_t\,\frac{I}{I_1}\left(\frac{M_1}{M}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(114)}$$

The conditions assumed for developing Equation (114) do not exactly fit those encountered in practice, but they do lead to a clear presentation of the effect of the moment and moment of inertia on the chord areas of a truss. The economy in truss web members, structrual and design details, and handling is not represented in Equation (114). The chord areas of all the trusses listed in Table 7 can be checked one against another with this formula. The results agree closely.

If the depth of the hypothetical truss and the unit stress in the chords are kept constant, it can be demonstrated that the moments of inertia vary directly with the moments. Thus, in a constant depth stiffening (suspension bridge) truss, the average moment and the average moment of inertia occur at the point, in theory at least.

Fig. 16 shows the curves of maximum moments for Design III and the hinged design. The loading positions for maximum moments as given by the author proved to be correct for Design III, which design effects a 16% saving in total moment area when compared to the hinged design. The saving

in truss chord material cannot be predicted from the saving in moment area. The average of the side-span moments is 39% greater than the average of the main span moments. The assumed relation, $I_1 = 1.25\ I$, is checked fairly well by the computed depths of the main and side span trusses (Table 7).

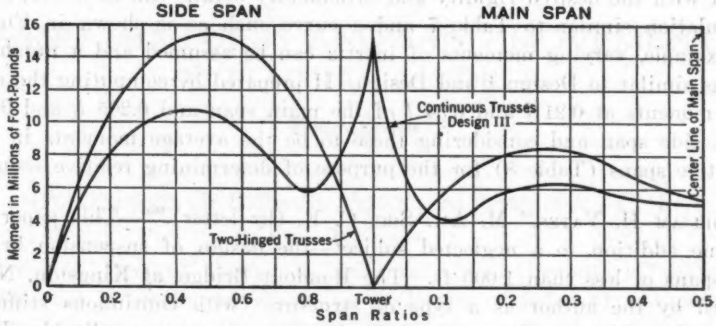

FIG. 16.—COMPARISON OF MAXIMUM MOMENTS FOR LIVE LOAD AND TEMPERATURE IN CONTINUOUS AND TWO-HINGED STIFFENING TRUSSES.

The average of maximum moments can be determined by computing the maximum moments at the points along the truss shown in Table 8. These points are practically identical for the three continuous designs.

TABLE 8.—SHOWING LOCATIONS ALONG TRUSS WHERE THE AVERAGE OF MAXIMUM MOMENTS (SHOWN IN TABLE 7) OCCUR. POINTS ALONG THE MAIN SPAN ARE SYMMETRICAL ABOUT THE CENTER LINE AND ONLY ONE-HALF ARE SHOWN.

Design	Span	Distance from left end
Hinged	Left side Main	$0.203 l_1$ and $0.790 l_1$ $0.124 l$ and $0.416 l$
I	Left side Main	$0.205 l_1$, $0.695 l_1$, and $0.915 l_1$ $0.060 l$, $0.205 l$, and $0.3475 l$
II	Left side Main	$0.215 l_1$, $0.655 l_1$, and $0.900 l_1$ $0.0575 l$, $0.210 l$, and $0.3775 l$
III	Left side Main	$0.205 l_1$, $0.665 l_1$, and $0.905 l_1$ $0.0575 l$, $0.2175 l$, and $0.340 l$

Column (6), in Table 7, is determined by assuming sections having the approximate areas required, calculating the allowable stress in the compression chord, determining the ratio of gross to net area for the tension chord, and using these values for determining the gross chord area shown. This is called a theoretical area because the detail of the section, when determined, may provide excess area. It may be possible to improve any of the designs by changing the moments of inertia so as to obtain more nearly equal depths in the side and main spans.

The author's comparison between hinged and continuous stiffening trusses is misleading since he takes no account of the effect on truss depth of the moment of inertia and the corresponding maximum moments. Economy in

truss chord material depends on a number of factors and not solely or directly on the saving in moment area of one design over another. The data in Table 7 support this conclusion.

The truss properties for a structure afford maximum economy consistent with the desired rigidity and satisfactory details can be selected from a tabulation similar to Table 7 and a curve such as is shown in Fig. 14. For example, varying moments of inertia can be assumed and a number of designs similar to Design I and Design III prepared by computing the maximum monents at 0.21 l and 0.34 l of the main span and 0.205 l_1 and 0.67 l_1 of the side span and considering these to be the average moments in their respective spans (Table 8) for the purpose of determining relative economy.

William H. Yates,[56] M. Am. Soc. C. E. (by letter).[56a]—This paper is a welcome addition to a neglected subject—the design of suspension bridges with spans of less than 1 000 ft. The Rondout Bridge, at Kingston, N. Y., is cited by the author as a type of structure (with continuous stiffening trusses) for which, until now, the deflection theory was not applicable. When it was designed in 1920 a popular impression that suspension bridges deflected excessively, had to be overcome. Consequently, a design was proposed that would have only one-half the deflection customary at that time. This was done by making the stiffening trusses continuous between anchorages instead of designing the usual type with a hinge at each tower. In fact, the Kingston Bridge is an unusually rigid structure, due not only to the continuous stiffening trusses, but also to the heavy reinforced concrete deck which made the dead load stresses about double the live load stresses. With a lighter deck the reverse would be true. The width of this deck and its location about midway between the upper and lower chords of the stiffening truss permitted an independent horizontal wind truss about 35 ft in width or depth. This is about one-twentieth of the length of the channel span, which is 705 ft long.

This span is generally regarded as too short for the economical use of the suspension type. Since its completion in 1922 several other bridges of that type have been built with spans of about the same length. The Kingston Bridge, therefore, was a pioneer among short-span suspension bridges designed for modern heavy highway loading.

Many variations of live load were assumed in the design of the chords and web members of the continuous stiffening trusses to determine the maximum moments and shears. The formulas made available in this paper would have reduced the number of computations greatly.

[56] Civ. Engr., New York, N. Y.

[56a] Received by the Secretary September 19, 1934.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

SAND MIXTURES AND SAND MOVEMENT IN FLUVIAL MODELS

Discussion

BY LORENZ G. STRAUB, ASSOC. M. AM. SOC. C. E.

LORENZ G. STRAUB,[55] ASSOC. M. AM. SOC. C. E. (by letter).[55a]—There is so much that is not known about the mechanics of sediment transportation in open channels that the amount known is negligible in comparison. Captain Kramer's experimental investigations and the analysis of the results of his own experiments and those of other investigators are, therefore, to be welcomed by the profession, particularly in the field of experimental design of erodible-bed streams.

It is doubtless mostly the lack of knowledge of the mechanics of bed-load movement (as well as suspended-load movement) which frequently leads engineers of long experience in the art of river regulation to voice the opinion that each river is a law unto itself—that the knowledge gained in the study of the behavior of one river is of little value in predicting the occurrences in another. Not infrequently experienced river engineers compare a river to a human personality whose "character and peculiarities must be understood in order to make him manageable." In truth, however, in the case of the inanimate river at least, the statements are warranted only because of the inadequate knowledge of the invariable, although complex, natural laws governing the occurrences.

In reviewing literature which has appeared during the past few decades on the subject of bed-load movement of rivers, one finds many statements which appear to be contradictory and lead to confusion. More careful scrutiny, however, shows that at least in some instances the apparently contradictory statements are reconcilable. A typical example concerns the transportability of bed load; one group of investigators, including the late G. K. Gilbert, of the U. S. Geological Survey, has analyzed experimental data on the basis of stream velocity. Another group led primarily by the du Boys method of approach has adopted expressions which include tractive force as a primary function in the relation for rate of bed-load movement.

NOTE.—The paper by Hans Kramer, Assoc, M. Am. Soc. C. E., was published in April, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1934, by Messrs. John Leighly, Paul W. Thompson, and Gerard H. Matthes; September, 1934, by Messrs. R. H. Keays, and F. T. Mavis; November, 1934, by Messrs. V. V. Tchikoff, Morrough P. O'Brien and Bruce D. Rindlaub, and Herbert D. Vogel; and December, 1934, by Joseph B. Tiffany, Jr., Jun. Am. Soc. C. E., and Carl E. Bentzel, Esq.

[55] Associate Prof. of Hydraulics, and Head, Hydraulics Div., Univ. of Minnesota, Experimental Eng. Laboratories, Minneapolis, Minn.

[55a] Received by the Secretary December 11, 1934.

It may be shown by the use of suitable open-channel flow formulas as a connecting link that, when correctly formulated, the two types of equations must yield similar results.

At present, there is a definite gap between theoretical assumptions and actual occurrences. From a practical viewpoint this is unimportant provided the formulas theoretically derived give results corresponding to actual occurrences when suitable empirical coefficients are introduced. Thus, the du Boys equation assumes that a series of thin laminae of sediment moves smoothly over one another over the entire stream bed. As a matter of fact, observation reveals that, except for unusual flow conditions, the sedimentary material moves down stream in riffles which roll over and over in a manner comparable to the movement of sand dunes. Just down stream of the miniature dune there is virtually no motion of sediment particles, such particles being rolled up the dune only to fall motionless at the down-stream or leeward side. This type of sediment movement is far different from that assumed in setting up the du Boys theory. Nevertheless, for sediment with certain size limits (ranging possibly from something less than 1 mm to about 4 or 5 mm in diameter) observations have shown that the equation gives satisfactory results with regard to the quantity of material moved, provided satisfactory empirical coefficients are used. Therefore, for all practical purposes, the use of the equation is justifiable even if the motion occurences are different from those assumed in setting up the theory.

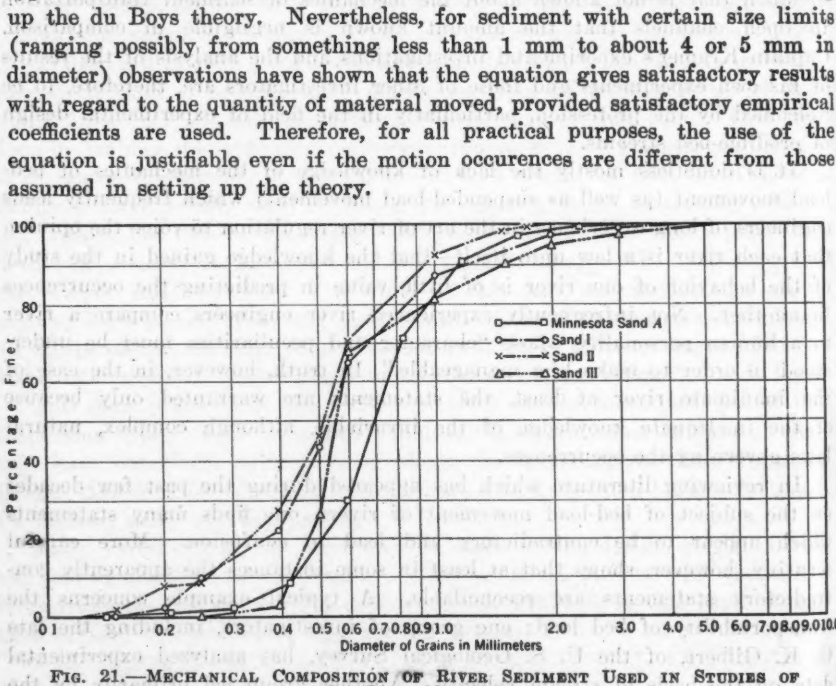


FIG. 21.—MECHANICAL COMPOSITION OF RIVER SEDIMENT USED IN STUDIES OF BED-LOAD MOVEMENT.

The author's Conclusion 1 concerning the "Law of Constant Critical Tractive Force," in which he states that "one and the same bed-load material under the usual moderate slopes invariably gives the same value for the lower critical tractive force," agrees very well with the writer's observations in laboratory experiments on a wide variety of river sands. However, there are a large number of investigators of note who have questioned the applicability of this law, particularly in the case of sediments made up of

particles ranging within wide size limits. It will be found that the acceptance of this law makes possible the formulation of equations which will closely define the bed load for wide variations in flow conditions.

In connection with a series of experiments being conducted at the University of Minnesota, Minneapolis, Minn., careful observations are being made of the nature of riffle formation and movement for different types of sediment. A typical example is here presented; the type of sand, at least as far as average size is concerned, is similar to the types used by the author. The material is typical river sand collected from the bed of one of the mid-Western streams. The mechanical composition is shown graphically by a semi-logarithmic plotting in Fig. 21, the author's Types I, II, and III, being superimposed for the purpose of comparison. Experiments are performed by maintaining the water-discharge constant and supplying bed-load material at the up-stream end of a channel at a given rate, the rate of feeding being maintained until complete equilibrium of all flow conditions is obtained in

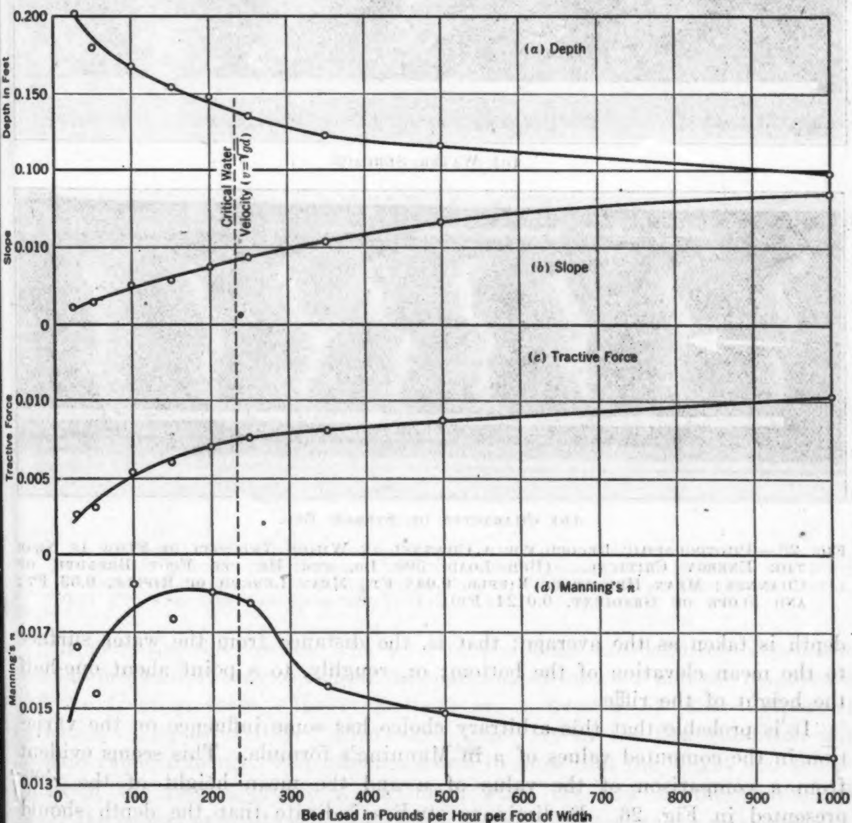
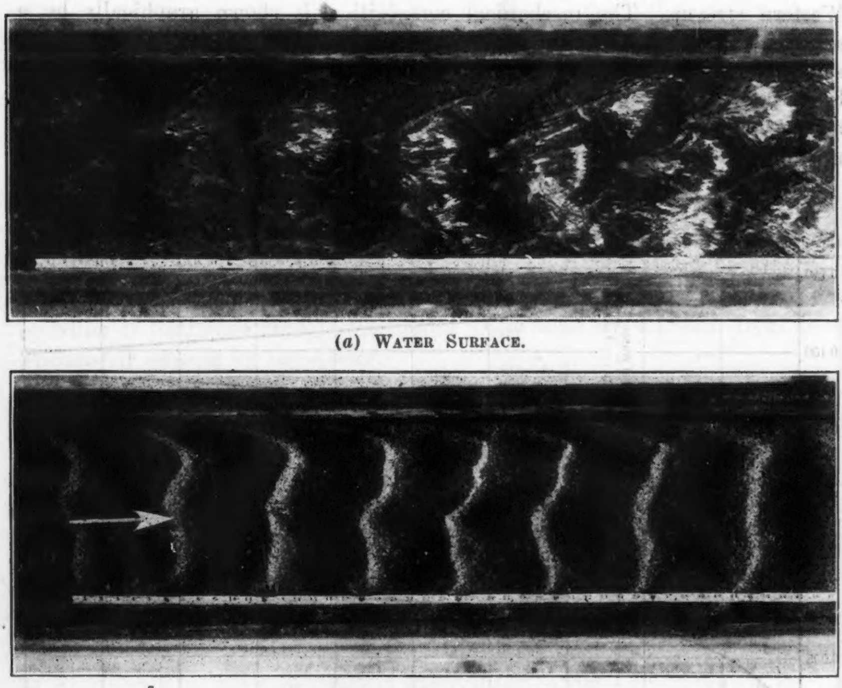


FIG. 22.—RESULTS OF MEASUREMENTS FOR TYPICAL SET OF EXPERIMENTS MADE TO DETERMINE CHARACTERISTICS OF BED-LOAD MOVEMENT. (DISCHARGE IS 0.30 CU FT PER SEC FOOT WIDTH OF CHANNEL FOR ALL MEASUREMENTS.)

the alluvial channel. Fig. 22 shows the results of measurements for a typical set of runs in which the water discharge was maintained at 0.30 cu ft per sec per ft width of channel. Figs. 23, 24, and 25 show photographic records of the riffle formation on the stream bottom for three different flow conditions, and the corresponding water surfaces; the water discharge (0.20 cu ft per sec) is the same for the three cases. The plotted points are the results of individual experimental determinations. Because of the occurrence of undulations in the elevation of the stream bottom some question arises as to the proper choice of depth. In the analysis presented in Fig. 22, the



(*a*) WATER SURFACE.

(*b*) CHARACTER OF STREAM BED.

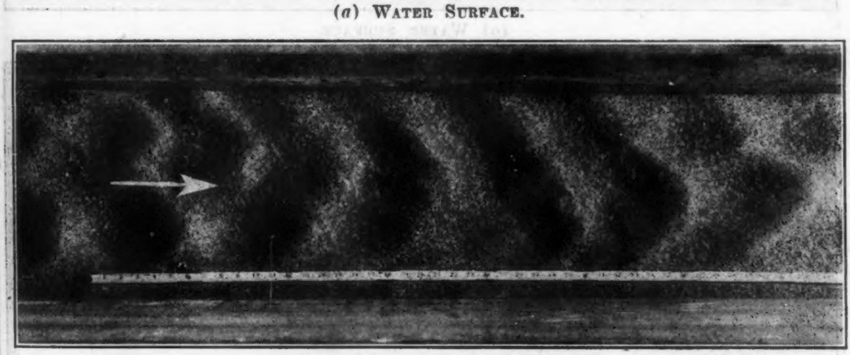
FIG. 23.—PHOTOGRAPHIC RECORD FOR A CHANNEL IN WHICH VELOCITY OF FLOW IS NEAR THE ENERGY CRITICAL. (BED LOAD, 300 LB. PER HR PER FOOT BREADTH OF CHANNEL; MEAN HEIGHT OF RIFFLE, 0.048 FT; MEAN LENGTH OF RIFFLE, 0.53 FT; AND SLOPE OF GRADIENT, 0.0124 FT).

depth is taken as the average; that is, the distance from the water surface to the mean elevation of the bottom; or, roughly, to a point about one-half the height of the riffle.

It is probable that this arbitrary choice has some influence on the variation in the computed values of n in Manning's formula. This seems evident from a comparison of the value of n and the mean height of the riffle presented in Fig. 26. Preliminary studies indicate that the depth should probably be taken as the distance from the water surface to a point practically at the elevation of the crest of the riffles, particularly when the riffles are

definitely of dune shape, with a flat up-stream slope and a steep down-stream slope; in the case of smooth undulations such as those indicated by the photograph presented in Fig. 24(*b*), the depth is probably more nearly the distance from the water surface to a point somewhat below the crest of the riffle. More studies should be made to determine the correct relation for analytical treatment. In any case, since an arbitrary value is taken for the depth corresponding to the average depth of water, it will be found that the roughness factor, *n*, in Manning's formula, increases with the relative height of the riffles.

(*a*) WATER SURFACE.

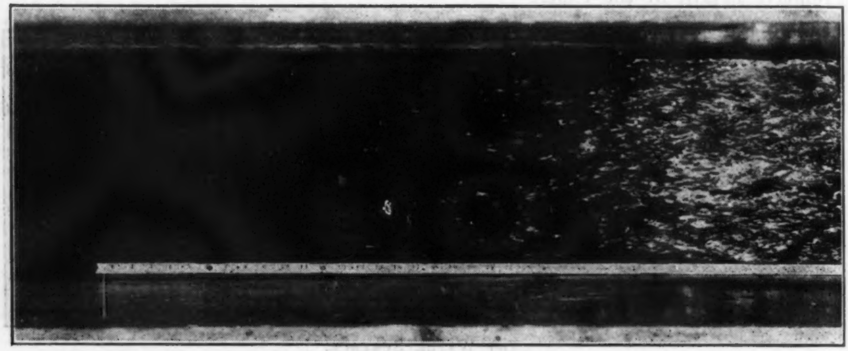


(*b*) CHARACTER OF STREAM BED.

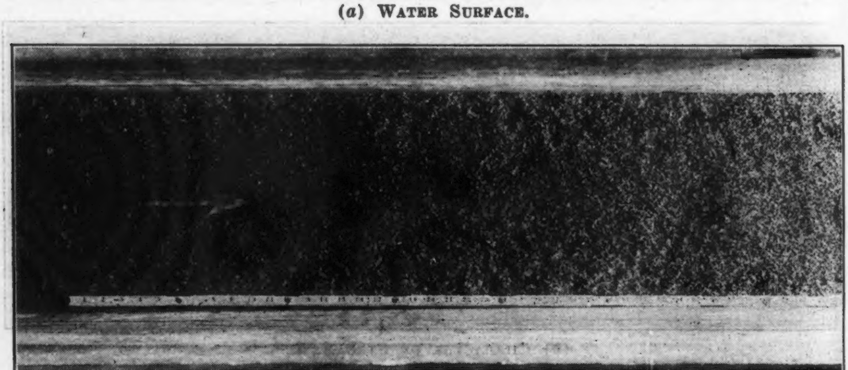
FIG. 24.—PHOTOGRAPHIC RECORD FOR A CHANNEL IN WHICH VELOCITY OF FLOW IS CONSIDERABLY BEYOND THE ENERGY CRITICAL. (BED LOAD, 450 LB PER HR. PER FOOT BREADTH OF CHANNEL; MEAN HEIGHT OF RIFFLE, 0.047 FT; MEAN LENGTH OF RIFFLE, 0.61 FT; AND SLOPE OF GRADIENT, 0.0143 FT.)

Apparently, the size and character of the riffles are definitely dependent on the mechanical composition of the sedimentary material and the magnitude of the traction force. However, there are a number of other influential criteria. Unquestionably, the so-called kineticity of the flow condition has considerable influence. Experiments seems to show that the height and sharpness of the riffle approach a maximum when the flow in the channel is near the energy critical (where the water velocity is equal to the wave velocity, $v = \sqrt{gd}$), as indicated by Fig. 23(*b*); for higher velocities the undulations are less rugged, and for still greater velocities relative to the energy critical

the bed appears to become smoother until, finally, the riffles disappear entirely, leaving a smooth stream bed, as indicated by Fig. 25(*b*). Unfortunately, to the present time the experimental data available do not permit definite conclusions concerning the relative influence of magnitude of traction force and degree of kineticity on the character of the riffle formation. Most experiments that have been performed by various investigators to the present time have either been almost entirely for streaming flow, such as the author's, or for shooting flow, such as Gilbert's experiments,



(*a*) WATER SURFACE.



(*b*) CHARACTER OF STREAM BED.

FIG. 25.—PHOTOGRAPHIC RECORD FOR A CHANNEL, IN WHICH VELOCITY OF FLOW IS GREATLY BEYOND THE ENERGY CRITICAL. (BED LOAD, 800 LB PER HR. PER FT BREADTH OF CHANNEL; NO RIFFLES OBSERVED; AND SLOPE OF GRADIENT, 0.0205 FT.)

although some of the studies of the latter include flow conditions with the velocity less than that for the energy critical.

The writer's observations indicate that the length of the riffles is much less subject to fluctuation with flow conditions than the height. However, much remains to be done to arrive at conclusions concerning the variation in riffle lengths. There is some indication that the length increases with the water discharge when the sediment load per unit breadth of channel remains constant.

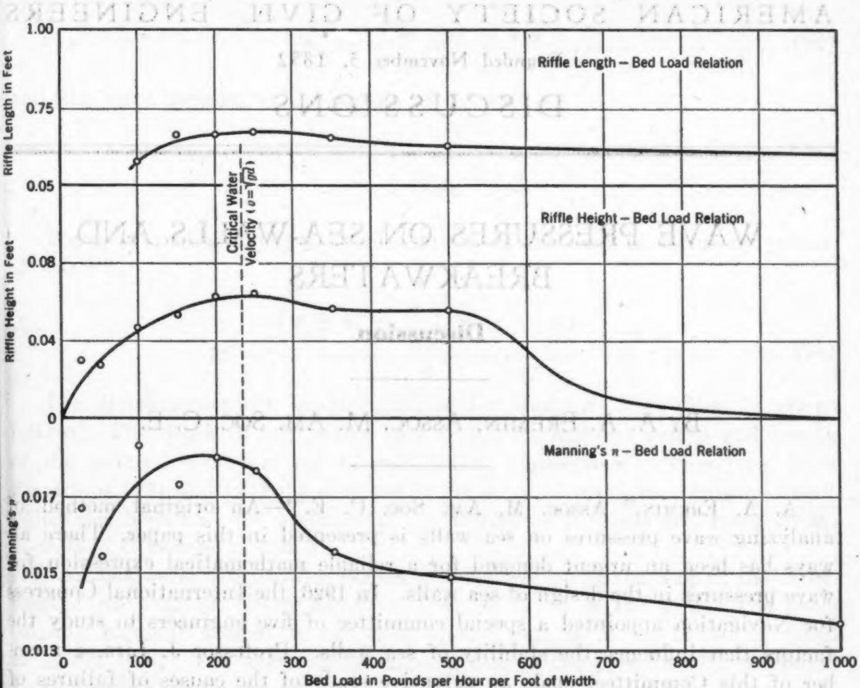


FIG. 26.—COMPARISON OF VALUES OF n IN MANNING'S FORMULA WITH MEAN HEIGHT OF RIFFLE FOR WATER DISCHARGE OF 0.030 CU FT PER SEC PER FOOT WIDTH OF CHANNEL AND VARIATION IN BED LOAD.

An analysis of river hydraulics problems by means of erodible-bed models will emphasize the necessity of further knowledge concerning the mechanics of bed-load movement. Almost invariably the occurrence of riffles in the model is so greatly out of proportion to those which obtain in the prototype that the results of the studies have grave limitations in their applicability to the prediction of the results to be obtained from projected control measures.

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DISCUSSIONS

WAVE PRESSURES ON SEA-WALLS AND BREAKWATERS

Discussion

BY A. A. EREMIN, ASSOC. M. AM. SOC. C. E.

A. A. EREMIN,[17] ASSOC. M. AM. SOC. C. E.[17a]—An original method of analyzing wave pressures on sea walls is presented in this paper. There always has been an urgent demand for a reliable mathematical expression for wave pressures in the design of sea walls. In 1926, the International Congress for Navigation appointed a special committee of five engineers to study the factors that influence the stability of sea walls. Professor J. Lira, a member of this Committee, made an extensive study of the causes of failures of walls at various seaports.[18] He found that the wave pressures computed by the method introduced in 1929 by G. Sainflou[19] are in agreement with the actual forces that caused the failures of the walls.

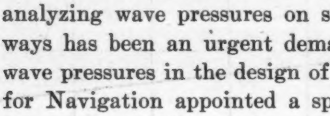

FIG. 11.—WAVE PRESSURE ON VERTICAL WALL ASSUMED BY SAINFLOU.

A coefficient of safety slightly greater than unity has been found sufficient in designing new sections of walls to replace sections that have failed. It is instructive to compare the pressures computed by Mr. Molitor in Fig. 7, with those computed by the Sainflou method. Fig. 11 defines wave pressures according to Sainflou's method, the notation being same as that used by the author. The point of maximum pressure, h_1, above still-water level, is given by the expression:

NOTE.—The paper by David A. Molitor, M. Am. Soc. C. E., was published in May, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: September, 1934, by Charles E. Fowler, Esq.; and December, 1934, by Charles T. Leeds, M. Am. Soc. C. E.

[17] Associate Bridge Designing Engr., Bridge Dept., State Highway Div. of Public Works, Sacramento, Calif.

[17a] Received by the Secretary November 28, 1934.

[18] *Le Genie Civil*, April 6, 1933.

[19] *Loc. cit.*, January 26, 1929.

$$h_1 = \frac{\pi h}{4 L} \coth \frac{2 \pi d}{L} \qquad (22)$$

and, the wave pressure at the bottom of the wall is,

$$p_b = \frac{h}{2 \cosh \frac{2 \pi d}{L}} \qquad (23)$$

The overturning moment at the bottom of the wall in terms of volume of water is:

$$M = \frac{\left(d + h_1 + \frac{h}{2}\right)^2 \times (d + p_b)}{6} - \frac{d^3}{6} \qquad (24)$$

The dimensions given by Mr. Molitor for Magann's Pier (Fig. 7) are as follows: The depth of still water is $d = 9.0$ ft; and the height and length of the reduced wave are 6.9 ft and 106 ft, respectively. Therefore, from Equations (22) and (23) the distance of the point of maximum pressure above still-water level and the pressure at the bottom of the wall are, respectively, $h_1 = 0.73$ ft and $p_b = 3.02$ ft.

The maximum overturning moment at the bottom of the wall in the direction of wave action, from Equation (24), is $M = 14\,150$ ft-lb. The overturning moment normal to the face of the wall is $M = 14\,150 \times 0.846 = 11\,971$ ft-lb. The overturning moment due to the pressure, P_o, computed by the author, is $M = 1\,580 \times 10 = 15\,800$ ft-lb, or, about 30% on the safe side as compared with that computed by Sainflou's method.

It is interesting to note also that Sainflou assumed the wave pressure to extend to the bottom of the wall (see Fig. 11).

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DISCUSSIONS

SECURITY FROM UNDER-SEEPAGE MASONRY DAMS ON EARTH FOUNDATIONS

Discussion

BY MESSRS. JOEL D. JUSTIN, AND LOUIS E. AYRES

JOEL D. JUSTIN,[40] M. AM. SOC. C. E. (by letter).[40a]—A thorough research is evidenced by this paper and to the data collected, the author has applied hard common sense and good judgment, demonstrating a thorough knowledge of sound design and construction.

That a vertical cut-off will be more effective in reducing the velocity of flow through foundation materials than an equivalent horizontal distance along the base, seems entirely logical. This is doubtless true in many cases, even if the contact of the base with the foundation material is insured by extensive and thorough grouting operations. One important reason is that, in actual cases, the foundation material is rarely, if ever, homogeneous, but occurs in strata and lenses of varying degrees of permeability. With a vertical cut-off, the various strata or lenses are penetrated, forcing the water to pass through the strata in a vertical direction (more or less), including some strata of more impervious material than that immediately below the base of the dam, thus decreasing the velocity of the seepage.

The fact that the author's weighted-creep criterion produces ratios lower than those of Bligh, may lead some to assume that it is less conservative than that of Bligh. Mr. Lane states that his weighted-creep ratios are somewhat less than one-half the ratios of Bligh. However, the application of the weighted-creep ratios given in the paper will frequently yield base widths greater than those obtained by using the Bligh coefficients. Actually, the author's method is more logical and gives results which are, in general, more conservative than would be obtained by the use of the Bligh coefficients. This is indicated by the data in Table 5.

NOTE.—The paper by E. W. Lane, M. Am. Soc. C. E., was published in September, 1934, *Proceedings*. Discusssion on this paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs. William P. Creager, and L. F. Harza.

[40] Cons. Engr., Philadelphia, Pa.

[40a] Received by the Secretary November 22, 1934.

It will be noted from Table 5 that, for a dam with a 30-ft head, founded on coarse sand, with a 60-ft cut-off, the author would obtain a base width of only 90 ft as against 240 ft, which would be obtained by the use of Bligh's method. In this case Mr. Lane appears to be less conservative than Bligh, but in the case of an over-flow dam, it is probable that the

TABLE 5.—COMPARISON OF REQUIRED BASE WIDTHS OF DAM BY LANE'S RATIO AND BLIGH'S RATIO; HEAD, 30 FEET.

Description	(a) DAM ON FINE SAND, OR SILT			(b) DAM ON COARSE SAND		
	Base Width, in Feet:		Ratio: Column (2) / Column (3)	Base Width, in Feet:		Ratio: Column (5) / Column (6)
	By Lane's coefficient	By Bligh's coefficient		By Lane's coefficient	By Bligh's coefficient	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Creep ratios........	8.5*	18	....	5*	12	....
Depths of Cut-Off, in Feet:						
0..................	765	540	1.42	450	360	1.25
15 (=0.5 H)..........	675	510	1.32	360	330	1.09
30 (=1.0 H)..........	585	480	1.22	270	300	0.90
60 (=2.0 H)..........	405	420	0.96	90	240	0.38

* Weight of horizontal creep, one-third.

base width required to secure a structure to smooth out the velocity of the water passing over the dam and to dissipate this velocity so that there will be no erosion, would become the controlling factor. Table 5 also shows the emphasis placed on vertical cut-offs by the author and indicates that it may frequently prove more economical to secure safety by deep cut-offs than by a great width of base.

Cut-offs of steel sheet-piling do not always give the same assurance of safety as a concrete cut-off may provide. If the foundation contains boulders, these piles may curl and split without this fact being detected at the time, with the result that the cut-off is quite largely ineffective. However, with most sandy materials, the use of an adequate high-pressure jet will permit driving the sheet-piling even to great depths with some assurance that the diaphram thus formed is continuous.

As pointed out in the paper, the weighted-creep method does not provide a complete criterion for securing dams on sands and soils safe against piping. Data on the foundation material of existing dams are far from precise. The occurrence of ledge rock or other impervious strata near the base of the dam, but at depths not reached by the cut-off, may affect, materially, the velocity of the water that may issue from the stream bed just down stream from the toe. Local conditions at or near the toe, such as lenses of coarse material, may cause a concentration of flow and may increase the velocity of the water at the egress to a point where there is danger of piping.

The author shows that it is necessary, for safety against piping, that the velocity of the water at the point of egress be so low that there is no danger of its carrying away the finer particles of the foundation material. For

expressing this requirement the writer has devised formulas[41] for the necessary length of the path of percolation, base width of dam, and necessary depth of cut-off, all based on Darcy's law. Thus, for the simple case of no cut-off, the formula is:

$$b = \frac{K H}{P V} \qquad (4)$$

in which, b is the necessary width of base; H, the head from head-water to stream bed at the toe of the dam; P, the porosity of the foundation material; K, the transmission constant of the material intercepted, or the weighted average transmission constant of the materials intercepted by the line of flow; and V, the safe velocity for the water at the point of egress, such that it will not move any of the finest foundation material (with a factor of safety applied).

Of course, Equation (4) is merely a straight derivation from Darcy's law. The formula is simple and is based on an entirely logical concept. The difficulty in applying it is in the choice of V, the safe velocity of the water at the point of egress. When sufficient experimental data become available, this formula and others dealing with the necessary depth of cut-off on a similar basis, will prove quite useful for aiding in the more precise analysis of the design of masonry dams on soils and sands, where relatively complete sub-surface investigations are made.

The use of the author's ratios will prove desirable as a basis for preliminary design, just as ratios are used in the preliminary design of masonry dams on ledge-rock foundations. Then, a thorough investigation at the site, including borings, test pits, mechanical analysis, and percolation tests should be made. In the case of important structures, model tests should be included, by all means. Finally, all these data should be interpreted in the light of experience, and the final design prepared accordingly.

LOUIS E. AYRES,[42] M. AM. SOC. C. E. (by letter).[42a]—The author has made a valuable contribution to the subject with which he deals. He has collected and critically analyzed much pertinent practical experience, and his comments thereon indicate a clear grasp of the design elements involved in dams on earth foundations. It is to be regretted that the "drawings that show dimensions and typical sections" of several of the dams investigated—at least, in sufficient number for illustration—could not have accompanied the printed text.[1a]

This paper presents a new set of arbitrary creep ratios which, on their face, appear to be less than one-half those heretofore recommended by Bligh and others. This apparent reduction has been accomplished by giving a greater weight to vertical than to horizontal creep. Obviously, Mr. Lane

[41] "Earth Dam Projects," by Joel D. Justin, M. Am. Soc. C. E., pp. 174–181, inclusive, John Wiley & Sons, Inc., 1932.

[42] Hydr. and Elec. Engr. (Ayres, Lewis, Norris & May), Ann Arbor, Mich.

[42a] Received by the Secretary November 27, 1934.

[1a] A lithographic set of the drawings may be purchased at a cost of 50 cents per copy from the Secretary of the Society.

might have presented equivalent creep restrictions, with values that would have appeared higher than those of Bligh, had he multiplied the vertical creep, instead of dividing the horizontal creep, by 3. This is merely stating that the new ratios are lower because of a change in definition and in emphasis. Actually, the new ratios permit of a reduction in the bottom width of a dam only if it is provided with a relatively deep up-stream cut-off; with a shallow cut-off, the bottom width required is greatly increased. The author's emphasis on the importance of adequate vertical creep is eminently proper. His recommended weighted ratios are an improvement on past usage and afford a conservative arbitrary standard, based on much experience.

A further logical step in the direction taken by Mr. Lane would be a recommendation to eliminate horizontal creep entirely from one's computations where it is practicable to obtain an adequate vertical cut-off. Many dams in recent years have been designed on this principle and many others, on porous soils, actually depend for their security solely on their vertical creep distances. The author quotes J. C. Oakes, M. Am. Soc. C. E., to the effect that under dams on piles in sand, where pumping is required during construction, it is "almost a certainty that there will be a space between the sand and the masonry." Whether or not, however, clearly defined open spaces exist, an average sand and gravel foundation will not offer much resistance to ordinary seepage along a smooth horizontal masonry surface; and if the seepage escapes readily at the toe, the structure becomes fairly well underdrained throughout. Certainly a dam's water-tightness under such conditions, if not its security, depends mainly on the up-stream cut-off; and the assumed advantage of horizontal creep has largely disappeared.

The author's investigations disclosed "many dams with a high proportion of vertical creep." These structures, in the main, are hollow dams of the multiple-arch and flat-slab types in some of which the supporting abutments rest directly on the soil, or on piles; such types depend for a cut-off solely on a line of steel sheet-piling driven along the up-stream edge. Where there is no floor between the abutments of such dams there is no bottom along which water must creep, and the creep distance becomes, therefore, twice the length of the cut-off. Obviously, the security of such a structure depends on the adequacy of the cut-off, which must penetrate the foundation material to a depth sufficient to prevent the possibility of a blow-out.

Although one may admit, with the author, that such a design is putting "all one's eggs in one basket," the evidence of experience shows that "basket" to be quite a reliable one. The author states that "not a single case was found in which piping unquestionably took place under a deep cut-off." He reports one failure because "one of the steel sheet-piles of the up-stream cut-off was pushed down far beyond its proper position, leaving an opening in the cut-off"; a second failure was due to "defective construction"; and a third came from "a blow-out directly through the foundation material from a porous layer beneath the impervious layer on which the dam was founded"; and, in this instance, the vertical creep distance was only 29 ft with a head of 33 ft.

Where there is no provision for horizontal creep under a dam, either because a floor is lacking, or the floor is underdrained, smaller creep distances than those shown in Table 3 would seem to be warranted. A range in depth of cut-off from a minimum of one-half the head in firm, homogeneous clay to about twice the head in sand would appear to be sufficient on the basis of much experience. An important caution in this connection is that proper depths may not be determined arbitrarily, but only on the evidence of an adequate soil investigation carried well below the bottom of the cut-off.

In his computation of vertical creep distance Mr. Lane considered "any contact with a slope of 1 on 1, or steeper," to be "vertical" contact. The most valuable part, by far, of the creep distance is the vertical path downward, along the up-stream face of the cut-off. The vertical path, upward, is certainly less valuable. For this reason the back-filling of clay, or other impervious material, against the up-stream face of a dam contributes to security. It is doubtful whether relatively shallow longitudinal ribs on the bottoms of dams, particularly in porous soil, are as effective per foot of depth as an equivalent depth added to the up-stream cut-off.

The author deals with the advantages of drainage under dams and aprons and he points out, very properly, the need for caution in location and for care in design. Many engineers are still reluctant to utilize drainage under dams on earth foundations and only a few seem to have taken full advantage of the judicious use of drains under such dams. There is a feeling that it is unwise to approach too closely to the up-stream cut-off with a drainage system and that down-stream and apron drainage only is consistent with safe design. There is an assumption of security in a long horizontal seepage surface and a fear of defects in a vertical cut-off. Does experience justify either the assumption or the fear? It is important, obviously, that drains be well designed and carefully constructed; that they be not so placed as to afford a short-cut between head and tail-water; nor so built as to be crushed under load or settlement, or become plugged or otherwise ineffective. Drainage under dams, in embankments and in connection with many other structures subject to upward pressure, affords a valuable tool in experienced hands, which has been used too sparingly.

Except under certain special conditions such as: (*a*) In some very fine sands or river silts where a "floated" dam may be required; or (*b*) in soil containing numerous large boulders or which, for other reasons, is difficult to penetrate with steel sheet-piling, a dam provided with an adequate cut-off and a reliable under-drainage system is a more secure and less expensive structure than a dam without drainage, depending on its weight to hold down upward pressure and a relatively long horizontal seepage surface to overcome percolation.

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DISCUSSIONS

UPLIFT AND SEEPAGE UNDER DAMS ON SAND

Discussion

BY MESSRS. JOEL D. JUSTIN, AND CHARLES TERZAGHI

JOEL D. JUSTIN,[23] M. AM. SOC. C. E. (by letter).[23a]—Darcy's law, as expressed in Equation (1) of this paper, may also be written $\frac{v}{K} = \frac{h}{l}$. Referring, furthermore, to Equation (6), F_c is shown to be approximately equal to 1 for the usual values of P and s. It should be borne in mind that F_c is the critical gradient at the point of egress and may be very different from the average hydraulic gradient. Hence, in addition to the manner in which the author has stated the equation, the critical condition for flotation at the toe may be stated as occurring when $\frac{v}{K} = F_c = 1$; or, it may be stated in words, as follows: The critical condition of flotation at the toe occurs when the velocity at the point of egress is approximately equal to the transmission constant for the given material.

The fact that $\frac{v}{K}$ is equal to $\frac{h}{l}$ enables the author to state that in "all mathematical or experimental investigations of base uplift pressure or flotation gradient at the toe, in homogeneous material, the transmission constant, K (Equation (1)), cancels out." As shown in this discussion, however, it would be equally feasible to eliminate $\frac{h}{l}$ from consideration and use the velocity and the transmission constant as the criterion.

On the basis of the author's equations, it might be assumed by the casual reader that all earth dams or all masonry dams on sand, of equal head and equal base width, on whatever kind of sand they are founded (as long as it is homogeneous) would be equally safe against the action of water passing under the dam, ignoring, for the sake of simplicity, any possible cut-offs.

NOTE.—The paper by L. F. Harza, M. Am. Soc. C. E., was published in September, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs. Edward Godfrey, and Hibbert M. Hill.

[23] Cons. Engr., Philadelphia, Pa.

[23a] Received by the Secretary November 20, 1934.

Such a conclusion, however, would not be justified because the "escape gradient", or the hydraulic gradient at the point of egress may bear practically no relation to the average hydraulic gradient. In actual cases, this "escape gradient" may be, and usually is, determined by local conditions at and near the toe.

By the method advanced in this paper, the quantity of leakage and its velocity would become largely an economic matter. With the statement as circumscribed by the author, the writer is in entire agreement. However, in any actual case, there are other factors which may make the quantity of discharge and its velocity through the foundation material, a matter of the most vital importance to the safety of the structure. Some of these factors have not been sufficiently emphasized.

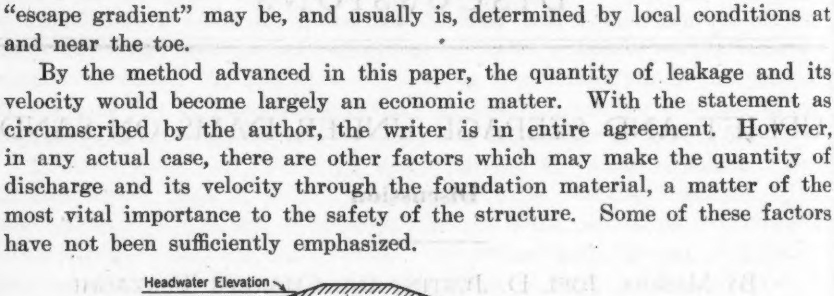

FIG. 19.—DAM OF IMPERVIOUS MATERIAL ON FOUNDATION OF VERY PERVIOUS MATERIAL OF GREAT DEPTH WITH LOW NATURAL GROUND-WATER LEVEL ON STEEP SLOPE.

Thus, for illustrative purposes, two extreme cases will be considered. In Fig. 19, the dam is assumed to be built of entirely impervious material on a foundation of very pervious material. The foundation material is entirely homogeneous and of great depth and the ground-water, before building the dam, is at a considerable depth and on a relatively steep slope; that is, due to the topography and geology of the site, the foundation is free draining.

After the construction of the dam there is a large amount of seepage under it, due to the pervious nature of the material, but, because the foundation has sufficient carrying capacity, this seepage is conducted away from the site without any of it appearing at the surface of the ground near the toe.

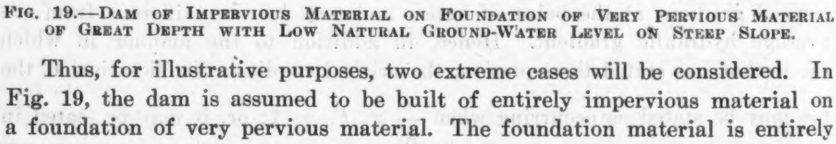

FIG. 20.—DAM OF IMPERVIOUS MATERIAL ON FOUNDATION OF VERY PERVIOUS MATERIAL AS IN FIG. 19, BUT WITH LEDGE ROCK NEAR SURFACE AND NATURAL GROUND-WATER SURFACE NEAR GROUND LEVEL AND ON FLAT SLOPE.

In Fig. 20, the same conditions occur as in Fig. 19, except that ledge rock or other impervious strata occur relatively near the ground surface. The natural ground-water, before the building of the dam, is also near the surface and is on a relatively flat slope. When the dam in Fig. 20 is built and the head-water is raised, water issues from the ground surface near

the toe of the dam. While Figs. 19 and 20 are entirely hypothetical, the conditions outlined measurably approach conditions at actual dams within the writer's experience.

There is no question as to the safety of the dam in Fig. 19 against any damage by the seepage of water under it, although the quantity of seepage carried away through the foundation may be considerable. On the other hand, the safety of the dam in Fig. 20 may be open to serious question. The "escape gradient" in Fig. 20 is evidently extremely steep, and its prediction would be difficult.

Attention is called again to the fact that this "escape gradient" may have practically no relationship to the average hydraulic gradient. In any actual case, conditions at and near the toe would determine very largely the actual "escape gradient". If the foundation material is extremely pervious (thus having a high transmission constant), the quantity of water carried would be large. Due to variations in the material, the flow may become concentrated and may issue from the ground below the toe with sufficient velocity to carry away the finer particles, thus starting the phenomenon commonly known as piping. In Fig. 20 conditions are favorable for the existence of a steep "escape gradient" and the possibility of "piping" is a real hazard, whereas in Fig. 19 there is no possibility of "piping".

As the "escape gradient" has almost no necessary relation to the average hydraulic gradient in most practical cases, it does not appear that safety against "flotation" or "piping" can be fully secured by a requirement that the base of the dam or that the length of the path of percolation shall be some given number of times the head, regardless of foundation conditions.

At any dam site where the foundation is sufficiently pervious so that large discharges through the foundation and high velocities may be anticipated, the possibility of "piping" requires careful consideration, and the same applies to earth dams built of very pervious material.

Thus, in the case of one dam built entirely of sand without a cut-off, it was considered that the anticipated seepage would not be of sufficient economic importance to justify an effective cut-off or diaphram. The anticipated discharge distributed throughout the cross-section gave a velocity low enough so that it did not appear that there was danger of moving any of the material. When the dam was first placed in service the actual seepage was about 20 cu ft per sec, which was a little less than that computed in advance of construction. However, the actual seepage was concentrated to a large extent which resulted in such high velocities at the egress that it carried with it the finer material from the surface. For a time it looked as if the dam might fail due to "piping" through the foundation at one of the abutments. Fortunately, the discharge finally worked back into gravel, too coarse to be moved. In the course of time, the seepage decreased considerably due, most likely, to the formation of a filter scum.

The possible effect of non-homogeneity in the material of the foundation and abutments, and the effect of ledge rock close to the surface causing a steep "escape gradient", had not (the event proved) been given sufficient con-

sideration. In the case of a similar sand dam, built on a deep sand foundation, there was no trouble, and practically no seepage at the toe.

When considering the possible effect of seepage under dams on pervious foundations, it is of prime importance to the safety of the dam to consider the quantity of such seepage, its velocity through the material, and its possible velocity on leaving the material. In this connection the efficacy of cut-offs and of up-stream blankets merits consideration. Sometimes the natural blanket formed by a relatively thin layer of top soil will have great effect in reducing velocity and the amount of seepage. In at least one case, near disaster followed the removal of a part of such a natural blanket.

In cases where the escape of a large amount of seepage at the toes is anticipated, some kind of drainage system is indicated with the drains partly clogged by gravel and stone so as to form inverted filters, thus preventing any of the foundation material from being carried through. Heavy rock fills in the down-stream part of an earth dam with the foundation protected by an inverted filter may perform this function.

CHARLES TERZAGHI,[24] M. AM. SOC. C. E. (by letter).[24a]—In reading Mr. Harza's stimulating paper the writer notes with satisfaction that the escape gradient is beginning to receive the attention which it has deserved for a long time. Since the writer has used what Mr. Harza calls the "rational approach" to the problem for more than ten years, he wishes to call attention to some of the difficulties connected with a rational treatment of the problem and to the methods for handling them.

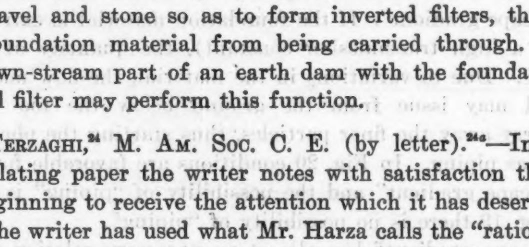

FIG. 21.

In 1922, when the writer derived[25] the equation for the flotation gradient (Equation (6) of the paper), he attempted at once to determine by experiment to what extent this factor can be used as a criterion for the safety of dams against piping. The results of experiments with perfectly homogenous sands showed that the escape gradient at which piping occurred is practically equal to the flotation gradient. Thus, if the base of the dam rests on the horizontal surface of the sand without any sheet-piles intercepting the flow of seepage, as in the case shown in Fig. 5 of the paper, the escape gradient is equal to

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[24] Dr. Ing.; Prof., Technische Hochschule, Vienna, Austria.

[24a] Received by the Secretary November 28, 1934.

[25] "Der Grundbruch an Staumauern und seine Verhütung," von Charles Terzaghi, M. Am. Soc. C. E., *Die Wasserkraft*, December 15, 1922.

infinity irrespective of the creep-head ratio, $\frac{L}{H}$. Hence, if the escape and the flotation gradient are really identical, piping should occur at any hydraulic head. This actually took place, and, as a consequence, the writer was compelled to establish the base of his flat-bottom model dams at a certain depth below the surface, in order to be able to produce any hydraulic head at all.

For a single row of sheet-piles with a depth, d, the escape gradient is approximately $\frac{2}{3}\frac{H}{L} = \frac{2}{3}\frac{H}{2d}$. Piping occurred in the writer's tests at an escape gradient which was a trifle less than the flotation gradient. The results of some of these tests are presented elsewhere.[26] The fact that the critical escape gradient for single rows of sheet-piles is slightly below the flotation gradient has also been explained by the writer.[27] However, for practical purposes, the difference can be neglected.

In order to demonstrate the practical consequences of these facts, consider the factor of safety against piping for the structure shown in Fig. 21. If a dam with such a cross-section is built on a perfectly homogeneous bed of sand, with no open space between the base and the sand, the escape gradient can easily be determined from the flow-net and expressed by a fraction, thus:

$$G = \frac{H}{\sigma\,(d_1 + d_2 + b + d_3 + d_4)} \qquad (16)$$

in which, σ is a "shape factor" the value of which depends on the details of the section as shown by the formula. The shape factor represents the ratio between the average hydraulic gradient along the line of creep and the escape gradient. If the width, b, is not excessive compared with the depth of the sheet-piles, the shape factor is not far from unity. However, for unfavorable cross-sections, it can be very much smaller, and in the extreme case represented by Fig. 5 in Mr. Harza's paper it becomes equal to zero. The flotation gradient is $(1 - P)\,(s - 1)$. Hence, the theoretical factor of safety against piping is:

$$f = (1 - P)\,(s - 1)\,\sigma\,\frac{d_1 + d_2 + b + d_3 + d_4}{H} \qquad (17)$$

The product, $(1 - P)\,(s - 1)$, has an average value of unity. The last factor represents the "percolation coefficient" of Bligh. Thus, the following conclusion can be drawn from Equation (17): For a given percolation coefficient, the factor of safety of a dam supported by a perfectly uniform sand depends on the value of the shape-factor, σ, which varies between 0 and more than 1.5.

In contrast to the assumptions on which Equation (17) is based, natural soil deposits are always more or less stratified and their average coefficient of permeability, k_v, in a vertical direction is always much smaller than the

[26] "Modern Conceptions Concerning Foundation Engineering," by Charles Terzaghi, M. Am. Soc. C. E., *Journal*, Boston Soc. of Civ. Engrs., December, 1925, Fig. 17.

[27] "Erdbaumechanik," von Charles Terzaghi, M. Am. Soc. C. E., Vienna, 1925.

coefficient, k_h, in a horizontal direction. From a great number of permeability tests made in connection with the design of dams the writer has reached the conclusion that the value of the ratio, $\beta = \frac{k_h}{k_v}$, can range between any value from about 2 to more than 10. This important and decidedly disagreeable fact prevented the writer from attempting to establish any general rules concerning safety against piping. If the water percolates through the non-homogeneous deposit in a vertical direction, the hydraulic gradient required for maintaining a definite speed of flow is β times greater than that for the flow with the same speed in a horizontal direction. Hence, the line of creep in a vertical direction has β times more "weight" than that in a horizontal direction. Taking this fact into consideration, the factor of safety against piping, in a non-homogeneous soil, is obtained from Equation (17), in the following form:

$$f = (1 - P)(1 - s)\,\sigma\,\frac{d_1 + d_2 + \frac{1}{\beta}\,b + d_3 + d_4}{H} \qquad \text{.......(18)}$$

The last factor in Equation (18) represents the "weighted creep" factor discussed in the paper[28] entitled "Security from Under-Seepage—Masonry Dams on Earth Foundations," by E. W. Lane, M. Am. Soc. C. E. Mr. Lane assumes that β is equal to 3. Equation (18) shows that the shape factor, varying between 0 and more than 1.5, may have an enormous influence on the factor of safety of the dam. This formula also shows that, for the same cross-section of the foundation of the dam, the weighted creep depends on the value of β, which may vary between 2 and 10. For this reason the writer welcomes Mr. Harza's suggestion that the analysis of existing dams be extended to include the influence of the cross-section of the foundation on the escape gradient and other vital factors.

If the soil profile is relatively simple, the ratio, β, can be estimated from the results of permeability tests. In this case the escape gradient can be determined from the known laws of the flow of water through non-homogeneous soils. According to methods developed for this purpose, the non-homogeneous material is replaced by a homogeneous material with a coefficient of permeability equal to $\sqrt{k_v\,k_h}$; and the horizontal scale of the section is reduced by $\sqrt{\frac{k_v}{k_h}}$. Although the writer always makes related investigations graphically, they may also be made by means of the electric analogy method proposed by Mr. Harza.

In most cases the failure of dams begins with the formation of springs which discharge a mixture of soil and water. For this reason, as soon as the writer recognized the importance of the flotation gradient, he concentrated his attention on the practical possibilities connected with reversed filters.[25] Since the effect of a filter on the factor of safety of a dam cannot be estimated by means of the formula for the flotation gradient, he was

[28] *Proceedings*, Am. Soc. C. E., September, 1934, p. 929.

obliged to develop a theory for computing the forces which are exerted by the seepage water upon the soil beneath the filter. Let i = the hydraulic gradient in the direction of the lines of flow; s_1 = the specific gravity of water; and p_w = the force exerted by the flowing water in the direction of the lines of flow. Between these quantities the following relation was found:[25]

$$p_w = i\ s_1 \quad \text{.........................(19)}$$

Mr. Harza's Conclusion 3 expresses Equation (19) in words.

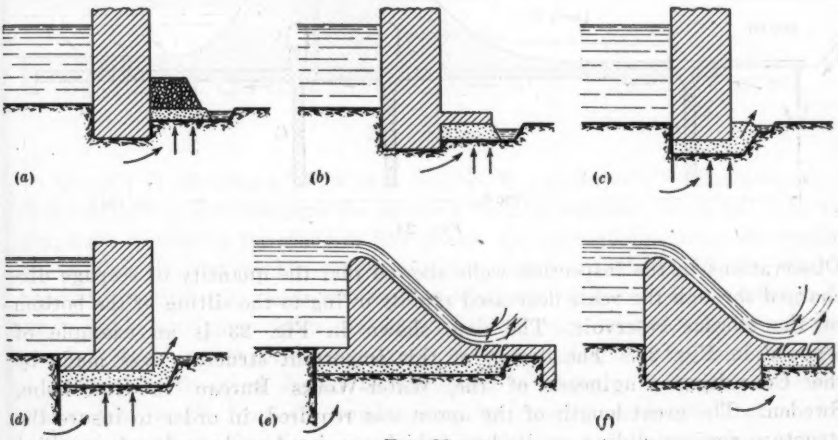

FIG. 22.

One of the essential requirements for making the filter effective is to prevent the flotation of the covered soil, by means of superimposed loads or by

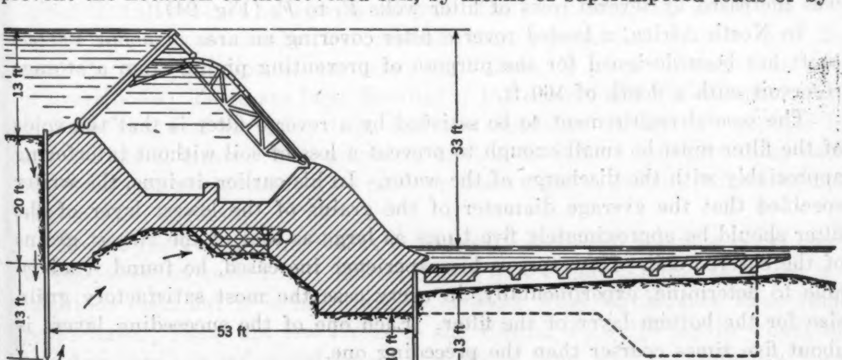

FIG. 23.

rigid obstacles. The results of tests related to this problem were presented in the paper cited previously.[26] Various possibilities for fulfilling this require-

[29] "Die Halleiner Wasserkraftstufe im Zuge du Salzachregulierung," von J. Pfletschinger, *Die Wasserwirtschaft*, June 5, 1929.

ment are indicated in Fig. 22, which is reproduced from a priority claim filed by the writer at the Austrian Patent Office in 1922. Figs. 23 and 24 illustrate the practical application of the principles embodied in Fig. 22. Fig. 23 represents a section through a bear-trap weir.[29] The surcharge consists of the weight of the water enclosed within the movable part of the construction.

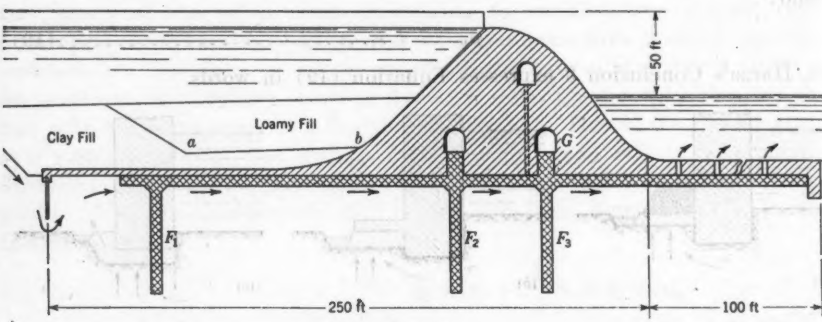

FIG. 24.

Observations in the inspection wells showed that the quantity of seepage discharged through the vents decreased rapidly owing to the silting of the bottom of the storage reservoir. The dam shown in Fig. 23 is an example of Type *d* in Fig. 22. The design of this important structure was made by the Consulting Engineers of the Water-Works Bureau in Stockholm, Sweden. The great length of the apron was required, in order to insure the structure against sliding on its base which consisted exclusively of stratified clay to a great depth. The surcharge is produced by the weight of the water above the rear apron and by that of the dam itself. The efficiency of the filter was increased by several rows of filter wells F_1 to F_3 (Fig. 24).

In North Africa, a loaded reverse filter covering an area of about 400 000 sq ft has been designed for the purpose of preventing piping from a storage reservoir with a depth of 160 ft.

The second requirement to be satisfied by a reverse filter is that the voids of the filter must be small enough to prevent a loss of soil without interfering appreciably with the discharge of the water. In his earlier designs, the writer specified that the average diameter of the grains of the lowest layer of the filter should be approximately five times as large as that of the largest grains of the covered soil. However, as his experience increased, he found it advisable to determine, experimentally, in every case the most satisfactory grain size for the bottom layer of the filter. Each one of the succeeding layers is about five times coarser than the preceding one.

Conclusion 11 of Mr. Harza's paper certainly should be emphasized. Owing to the great number of purely empirical factors involved, any theory of dam design urgently requires supplementing by the results of observations on existing structures.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

AN ASYMMETRIC PROBABILITY FUNCTION

Discussion

BY MESSRS. GORDON R. WILLIAMS, AND H. ALDEN FOSTER.

GORDON R. WILLIAMS,[22] JUN. AM. SOC. C. E. (by letter).[22a]—The purpose of this discussion is to compare the author's "totally bounded function" with the graphical method in the study of flow peaks. To the mathematician the graphical method is inexcusably crude, but the engineer finds in its use certain practical features that may overshadow what the mathematician considers its undesirable attributes from his standpoint of pure science. It is not proposed to delve into theory herein except to state that it is questionable whether a probability function should be applied to a study of natural phenomena which may result from the combination by summation, multiplication, and cyclical variation of various causes and conditions in a more or less indeterminate and discontinuous sequence.

Any purely mathematical treatment of stream-flow data presumes at the outset that all the data are equally reliable. Unfortunately, this is not the case. Early records are usually less reliable than present-day records, and the accurate determination of extremely high flows is more difficult than that of lower flows. These are facts familiar to the experienced hydrographer. The probable range of variation from the true discharge cannot be computed by any statistical means. If a graphical method is used to interpret these data, the curve can be shifted to give more or less weight to certain points, depending on their accuracy.

To use the author's method it is necessary to list all the flow peaks down to the lowest on record. Of course, it is a minor consideration that this is a laborious task, but it is important to consider what significance these small peaks have. There are few rivers in the United States to-day in which the low flows are not influenced by some works of Man, and, consequently, these flows would be so modified, at least in magnitude, as to have no relation to the frequency and magnitude of the higher flows. There may also be certain

NOTE.—The paper by J. J. Slade, Esq., was published in October, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[22] Asst. Engr., Water Resources Branch, U. S. Geological Survey, Washington, D. C.

[22a] Received by the Secretary November 10, 1934.

peculiarities of the natural drainage area and river valley which would make it doubtful whether one relation would suffice for the full range of flows. Most graphical methods do not attempt to utilize flows below the so-called "one-year flood"; that is, the smallest of the annual maximum floods.

After all the flows have been listed, it is necessary to assign an upper limit, which the author states must be determined by some extra-statistical means. The determination of this upper limit is a matter which is receiving careful consideration from many leading hydraulic engineers to-day. If it is possible to determine this limit accurately, the graphical method as well as the statistical method will benefit. For the graphical method, the upper limit will furnish an asymptote that will greatly increase the accuracy of this method, in the range between the observed data and the limit.

From studies of relatively long records it appears that any method that utilizes a record of less than 30 or 40 years will have to be adjusted for long climatic cycles. For example, any study that utilized the records of the period, 1914 to 1934, would lead in many places to a large under-estimate of the flood potentialities of the region. Phenomena that are influenced by such climatic cycles cannot be studied by mathematical means alone.

All the methods for analyzing flood and run-off phenomena have weaknesses, but a solution of the problem will be obtained only through a combined consideration of the valuable features of each method.

H. ALDEN FOSTER,[23] M. AM. SOC. C. E. (by letter).[23a]—An excellent discussion of probability curves in general, and of the relative merits of the several types of curves which have been proposed, is contained in this paper. The author's criticisms of the Pearson curves are worthy of note, and should be helpful to any engineer who has occasion to use these curves in practical work. His comments on empirical curves and graphical methods are also significant.

Professor Slade presents a new formula for statistical analysis, and develops it mathematically, with numerical examples. This formula is proposed as a substitute for other curves (particularly those of Pearson) which have been in use for many years. He specifies several desiderata which any probability function should satisfy in order to be mathematically consistent and suitable for practical use. The proposed formula is shown to satisfy these requirements; but whether it is superior in these respects to the Pearson curves has not been clearly demonstrated. In the writer's opinion, all these desiderata are satisfied by Pearson's Type I and Type III curves, as well as can be desired for any practical engineering uses.

Requirement (*d*), that "the curve must be simple to apply", does not seem to be well substantiated from a practical standpoint. Doubtless, the proposed curve is easier to handle than some other types of probability functions; but it appears doubtful whether the average engineer would be able to make ready use of it, due to the mathematical operations involved.

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[23] Res. Engr., Parsons, Klapp, Brinckerhoff & Douglas, North Platte, Nebr.

[23a] Received by the Secretary, December 28, 1934.

As has been shown by the writer,[24] the practical application of the Pearson curves (Types I and III) can be made very simple due to the fact that, with these curves, the distance from the mean to any point on the duration curve is directly proportional to the coefficient of variation. This relation does not appear to hold for the author's formula—at least, he does not mention it. It was by virtue of this relation that the writer was able to prepare tables from which the Pearson curves could be plotted directly, as soon as the proper values of the coefficient of variation (*CV*) and coefficient of skew (*CS*) were determined.

The author has given (see Table 2) what appears to be a similar table for his curve, showing variations from the mean at several percentages-of-time, for dilfferent values of the coefficient of skew. This table is computed for a coefficient of variation, *CV*, equal to unity; but there is nothing to show that it can be used for any other value of *CV*.

Examination of Table 2 indicates that the author's "partly-bounded function" gives results very close to the Pearson Type III curve for percentages-of-time greater than 0.1%; but for more extreme values of the probability scale the proposed formula gives increasingly larger values of the variation than by the Pearson curve.

To apply the new formula to a practical engineering problem, as shown in the examples given by the author, requires the computation of several constants in addition to the coefficients of variation and skew. Computation of ordinates of the frequency curve can then be made directly, by means of Equation (11); but the solution of the duration curve requires the use of a table of the probability integral which, although in common use by statisticians, is not always available to the practicing engineer. The proper method of using the table, even if available, might easily lead the average engineer into difficulties.

The foregoing comments apply particularly to the "partly-bounded function" proposed by the author. The "totally-bounded function" appears to involve still greater difficulties from a practical standpoint.

The writer has attempted to examine this paper from the viewpoint of the practical engineer. In many respects, the paper reflects the attitude of the expert mathematician or statistician rather than that of the engineer. It is quite possible that the author's formula may have great potential value in general statistical analysis. The writer does not consider himself qualified to discuss it from that angle.

[24] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 158.

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DISCUSSIONS

EFFECT OF SECONDARY STRESSES UPON ULTIMATE STRENGTH

Discussion

BY MESSRS. C. H. SANDBERG, AND J. D. GEDO

C. H. SANDBERG,[21] ASSOC. M. AM. SOC. C. E. (by letter).[21a]—Until recent years the analyses of the problems concerning secondary stresses and indeterminate stresses have been treated in a rather theoretical and academic manner. It is gratifying, therefore, to read a paper that discusses this problem of secondary stresses from a practical standpoint.

One point not emphasized by the authors is the advantage of knowing the effects of secondary stresses in determining the necessity for the replacement of old truss spans. Many of the old, riveted, railroad truss spans, for example, were fabricated with absolute disregard of eccentricity of connections. When the theoretical stresses are computed for the members of such trusses the extreme fiber stresses are often found to be at, or beyond, the yield point.

In addition to secondary stresses and those caused by seriously eccentric connections, some old truss spans carry the live load directly on either the top or the bottom chords. This causes extremely high combined stresses, in some cases; and yet the members are apparently carrying these theoretically excessive loads without any dangerous over-stress. Thus, engineers acquainted with old riveted truss spans can verify from actual experience, the conclusions advanced in this paper. Of course, it is agreed that secondary stresses and similar stress effects cannot be ignored indiscriminately.

It is of interest to note the beneficial effect of the stiffener bar that was welded to the cover-plate of Test Column No. 4. This method of stiffening could be applied, readily and cheaply, to compression members of existing old truss spans. It is hoped that some further investigations and experimental research will be made along these lines.

NOTE.—The paper by John T. Parcel, M. Am. Soc. C. E., and Eldred B. Murer, Jun. Am. Soc. C. E., was published in November, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[21] Asst. Engr., Bridge Dept., A. T. & S. F. Ry. System, Chicago, Ill.

[21a] Received by the Secretary December 5, 1934.

J. D. GEDO,[22] ESQ. (by letter).[22a]—The importance of this lucid paper is that it evaluates the effect of secondary stresses upon the ultimate strength and thus furnishes considerable information in regard to the fixing of rational unit stresses.

The only disturbing element in the exposition is that the secondary stresses sometimes are ascribed to the deflection of the truss (see under headings, "Synopsis", and "Analysis of Problem"), and sometimes to the gussets. If the first version were correct, one would conclude that there are no bending moments in the truss represented in Fig. 22, which is carved from a single piece of steel and in which every joint is supported.

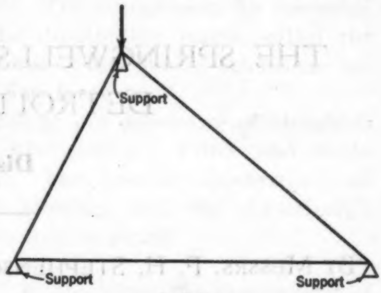

FIG. 22.

[22] Structural Designer, A. D. Crosett, New York, N. Y.

[22a] Received by the Secretary December 13, 1934.

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DISCUSSIONS

THE SPRINGWELLS FILTRATION PLANT, DETROIT, MICHIGAN

Discussion

By Messrs. F. H. Stephenson, and Robert Spurr Weston

F. H. Stephenson,[5] M. Am. Soc. C. E. (by letter).[5a]—Adequate and easily comprehended records of design and construction of large municipal projects, are not presented as frequently as they should be, and the author should be commended and congratulated for his contribution to this class of literature.

The writer's direct connection with the water supply of Detroit began with studies for the preparation of the Pratt report of 1919 mentioned by the author, and continued until after the beginning of operation of the Springwells Filtration Plant. With this background, the writer submits a few observations regarding the filtration of the water supply of Detroit.

The experimental filtration plant built in 1917 was designed primarily to give visual proof of the feasibility of filtering the existing water supply, and the improvement in appearance, taste, and purity which would be brought about by filtration. Its successful operation demonstrated that a satisfactory effluent could be produced with 2 hr. sedimentation and filtration at a normal rate of 160 000 000 gal daily per acre. The experimental plant constructed in 1925, was designed, as the author states, primarily to obtain information relating to the phenomena of mixing of chemical solutions with raw water, the formation of floc, and the settling of the treated water before it is conveyed to the filters.

The Springwells Plant, as built, retained the normal rate of filtration (160 000 000 gal daily per acre) and the 2-hr settling period found satisfactory in the older plant. Improved methods of introduction to, and guidance of, mixed water through the settling basins came as one of the results of lessons learned from the operation of the experimental settling basin. The

Note.—The paper by Eugene A. Hardin, M. Am. Soc. C. E., was published in November, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[5] Mount Vernon, N. Y.

[5a] Received by the Secretary December 12, 1934.

designers of the new plant set their goal as efficient mixing of chemicals and raw water, with optimum floc and settlement of the treated water, in order to lighten the work of the filters.

The addition, to the filter under-drain system, of a line of perforated pipe connecting the ends of the lateral pipes, should result in improved distribution of wash water at or near the filter walls. The arrangement for operating the pumps supplying filtered water to the distribution mains, called the shunt system, should result in reduced power costs. Where conditions are favorable, this method of operation should find favor.

The methods of unloading, storing, mixing, and delivering of chemicals or chemical solutions are modern and labor-saving. Filter and wash-water control are effective and efficient. The general appearance and architectural treatment of the plant are pleasing, and the municipality possesses a filtration system of which it may well be proud.

ROBERT SPURR WESTON,[6] M. AM. SOC. C. E. (by letter).[6a]—The account of the construction of the new Filtration Plant at Detroit, Mich., by its designer, which has followed the "Studies on the Washing of Rapid Filters",[7] by Messrs. Roberts Hulbert and Frank W. Herring, is of great importance to those interested in the progress of the art of water purification.

The plant at Detroit exemplifies the great improvement which has been evolved in the method of acquiring a water purification plant in the past forty years. In the Nineties one bought a stock filter and tried to make it work with the water supplied to it. Some good ready-made fits were obtained, and some of the purchases—Biddeford, Me.; East Providence, R. I.; Elmira, N. Y.; Atlanta, Ga., and others—are still in use, although some have been "re-tailored" in order to fit their waters better. Many others of these ready-made plants have been replaced.

Now, as exemplified at Detroit, one studies the water to be treated and builds a plant to fit it. As the author has shown, the first requisite is proper treatment — coagulation. With proper treatment, almost any sand bed produces good results; without it, the excellent one designed by the author might fail even if it is true that the proper treatment of the Detroit River water is not as difficult as that of unstored, colored or turbid waters, such as those of the Dismal Swamp, the Ohio River, or the Mississippi River.

For example, with inadequately treated water, it probably would be impracticable, even at Detroit, to use sand as coarse as that having an effective diameter of 0.55 mm, or beds only 20 in. thick; also, it would probably be impracticable to use a system of pipes for under-drains.

Naturally, the ideal system in mixers and basins for chemical treatment and coagulation is turbulent mixing after the addition of chemicals, to be followed, in turn, by slow mixing and subsidence, the velocity of flow in basins roughly conforming to a decreasing parabolic curve which reaches

[6] Cons. Engr. (Weston & Sampson), Boston, Mass.

[6a] Received by the Secretary December 21, 1934.

[7] *Journal* Am. Water Works Assoc., Vol. 21 (1929), p. 1445.

its minimum just before the water is applied to filters. In practice, this ideal is approached "step-wise", but invariably there are pipes and channels between mixers and basins and between basins and filters in which velocities are increased even to a degree that damages flocs formed previously under favorable conditions.

In this connection, the designers are to be congratulated in maintaining in so large a plant, influent and effluent velocities of 1.0 ft per sec, which, while high in comparison with the velocity of 2.8 ft per min through the basins, are lower than those obtaining in many other plants. Especially praiseworthy is the simple system of distributing the basin influent, although in cases where silt-bearing waters are treated there is something to be said in favor of diverting the inflowing water toward the bottom of the basin; also, where colored waters are treated, there is something in favor of the parabolic deflectors near the outlet weir, so successfully used at Baltimore, Md.

In common with some others, the practice of the writer has tended toward the use of filters with false bottoms because they provide a better distribution of wash water. This is because the conditions more nearly approach the ideal of multiple orifices discharging from a large tank. Experience shows that great damage is done to sand beds for which the wash-water distribution is uneven, and unbalanced hydraulic conditions seem to arise in certain designs of pipe manifolds which no reasonable thickness of gravel beneath the sand seems to correct, although Gore, in Canada, has used layers of cemented gravel for this correction, with reported success.

The recent design of the Mahoning Sanitary District filters,[8] using false bottoms supporting Wheeler pyramids and balls, and providing a large accessible water space beneath, seems to be superior to any pipe system. The writer's firm has recently used it at Braintree, Mass., with entire satisfaction.

The selection of wash-water rates of from 30 to 39 in. per min for a sand having an effective size of 0.50 to 0.55 mm, is interesting to the writer because, in 1913, he made experiments at Concord, Mass.,[9] using various types of sand. From these experiments there was derived an optimum velocity of 30 in. for a sand having an effective size of 0.55 mm, at which velocity it was estimated the sand would expand 39 per cent. In these experiments, however, no attention was given to variations in viscosity of wash-water due to variations in temperature.

In conclusion, the writer expresses his admiration, not only of the simplicity of the design, but of the fullness of the data presented in the paper, data truly useful to designers and users of other plants.

[8] *Engineering News-Record,* Vol. 111 (1933), pp. 317–321.
[9] *Engineering News,* Vol. 72, p. 22.

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DISCUSSIONS

EXPERIMENTS WITH CONCRETE IN TORSION

Discussion

By Messrs. H. J. Gilkey, and A. A. Eremin.

H. J. Gilkey,[16] M. Am. Soc. C. E. (by letter).[16a]—To the citations in this paper, of representative illustrations of torsional stresses in concrete structures, might be added many others. Further enumeration seems unnecessary because the frequency of their occurrence and the need for more information on the torsional behavior of concrete must be apparent to any thoughtful concrete designer.

The scarcity of experimental evidence on torsional behavior is apparent from the paucity of references cited under the heading, "Previous Investigations". Moreover, every previous investigation noted was conducted outside the United States. Any available additional evidence should constitute a desirable supplement to that presented.

During 1928 and 1929 Fredrik Vogt, Assoc. M. Am. Soc. C. E., and the writer conducted torsional tests on plain concrete in connection with the extensive auxiliary test program which accompanied the construction and testing of models of the Stevenson Creek and Gibson Arch Dams at the University of Colorado, by the U. S. Bureau of Reclamation in co-operation with the Committee on Arch Dam Investigation of Engineering Foundation. Some of these data reached publication in May, 1934,[17] and, therefore, were not available, in published form, until after Mr. Andersen's paper was in print. The tests included no specimens reinforced against torsional stresses, but some of the results supplement Mr. Andersen's tests on plain concrete in an interesting manner.

In this discussion it is proposed to avoid repeating, as far as is reasonably possible, material available in the published report, by makng specific references to it when needed.

Note.—The paper by Paul Andersen, Assoc. M. Am. Soc. C. E., was published in May, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1934, by E. Mirabelli, Assoc. M. Am. Soc. C. E.; October, 1934, by Messrs. Frank M. Russell, and Leslie Turner; and December, 1934, by A. W. Fischer, Esq.

[16] Prof. and Head of Dept. of Theoretical and Applied Mechanics, Iowa State Coll., Ames, Iowa.

[16a] Received by the Secretary December 4, 1934.

[17] "Tests of Models of Arch Dams and Auxiliary Concrete Tests Conducted by the Bureau of Reclamation at the University of Colorado," by Engineering Foundation, Committee on Arch Dam Investigation, Sub-Committee on Model Tests, Vol. II (May, 1934).

An arch dam is subjected to compressive, flexural, tensile, and shearing stresses. Torsion is, of course, a rotational shear. To interpret properly the test data obtained from the models, it was necessary to know the stress-strain properties of the material for each of these several types of stress. This involved, among other things, the determination of Poisson's ratio in compression, Young's modulus in compression, flexure, and tension, and the modulus of rigidity, or the corresponding shearing modulus. Thus, for two different concrete mixtures (that of the Stevenson Creek model and that of the Gibson model) there are available for direct comparison, results from all these several types of test. While the numbers of duplicate specimens are not large (except in the case of compression) the tests were well controlled and carefully executed, with the result that few values appear to be appreciably out of line.

When the Stevenson Creek tests were planned, it was expected that the testing of the model would be in the air-dry condition and it was necessary, therefore, to make the control program cover a range of cases which involved drying out in air for different lengths of time after various periods of moist curing. Later, it proved desirable to keep the model moist at all times up to and during the testing period, but the original varied curing program was followed through for the auxiliary model tests of Stevenson Creek Dam. This gives a wide range of strengths obtained by the variations in time and nature of curing and in the saturated and air-dry conditions at test. Thus, while the effect of varied curing conditions is only of secondary interest here, the different ages, curings, and test conditions do supply additional comparisons between torsional and other properties.

The test data appear in Table 5. It is to be noted that the details concerning curing and testing conditions (Column (2), Table 5) are given in Table 6. The torsional and flexural ultimate stresses in Column (4), Table 5, were computed by the elasticity formulas:

$$f = \frac{M\,c}{I} \quad \text{.........................}(35)$$

and,

$$v = \frac{M\,\rho_e}{J} \quad \text{.........................}(36)$$

in which, in addition to the notation in the paper, J = polar moment of inertia; c = distance from neutral axis to extreme fiber; and f = a unit fiber stress. Consequently, these values are approximate. Upton[18] developed a method for determining the true stresses for ultimate torsional and flexural loadings. The ultimate shearing stress at the extreme fibers in torsion reduces to approximately three-fourths of that given by Equation (36), and in flexure the ultimate stress is two-thirds the modulus of rupture. These values are given in Column (5), Table 5.

[18] "Materials of Construction", by G. B. Upton, pp. 52 and 78; or Johnson's "Materials of Construction", rewritten by M. O. Withey and James Aston (1919), Fifth Edition, John Wiley & Sons, New York, N. Y., pp. 23 and 28.

TABLE 5.—RESULTS OF TESTS IN COMPRESSION, FLEXURE, TENSION, AND TORSION

Line No.	Total age	Curing and testing condition*	Kind of test	Ultimate Strength, in Pounds per Square Inch: Approximate	Ultimate Strength, in Pounds per Square Inch: Adjusted (by Upton's method)	Modulus of elasticity, in millions of pounds per square inch	Unit Deformation, in Ten Thousandths of an Inch per Inch, at: 50% of ultimate load	Unit Deformation, in Ten Thousandths of an Inch per Inch, at: Ultimate load
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
				(a) Series I*				
1	7 days...	Standard...	Compression	1 780	...	1.83	5.00	16.50
2	7 days...	Standard...	Flexure.....	401	267	2.67	0.75	2.30
3	7 days...	Standard...	Tension....	234	...	2.10	0.34	0.78
4	7 days...	Standard...	Torsion....	284	213	0.97	1.49	3.63
5	28 days...	Standard...	Compression	3 200	...	2.69	6.60	20.50
6	28 days...	Standard...	Flexure.....	570	380	2.91	1.04	2.56
7	28 days...	Standard...	Tension....	275	...	2.65	0.48	1.10
8	28 days...	Standard...	Torsion....	413	310	0.99	2.10	5.01
9	3 months.	Standard...	Compression	4 250	...	2.91	7.50	22.00
10	3 months.	Standard...	Flexure.....	652	435	3.80	0.86	2.18
11	3 months.	Standard...	Tension....	329	...	3.20	0.44	1.04
12	3 months.	Standard...	Torsion....	447	335	1.33	1.49	5.09
13	1 year....	Standard...	Compression	4 850	...	3.32	7.60	25.00
14	1 year....	Standard...	Flexure.....	631	421	3.03	1.05	2.57
15	1 year....	Standard...	Tension....	333	...	3.05	0.55	1.42
16	1 year....	Standard...	Torsion....	504	378	1.58	1.68	4.32
17	1 year....	8W 4AD..	Compression	5 086	...	2.68	10.10	34.80
18	1 year....	8W 4AD..	Flexure.....	850	567	2.90	1.47	3.24
19	1 year....	8W 4AD..	Tension....	486	...	3.06	0.80	1.93
20	1 year....	8W 4AD..	Torsion....	639	479	1.45	2.16	4.86
21	1 year....	8W 4AW..	Compression	3 710	...	2.68	7.20	26.60
22	1 year....	8W 4AW..	Flexure.....	720	480	2.83	1.28	2.85
23	1 year....	8W 4AW..	Tension....	278	...	2.52	0.51	1.55
24	1 year....	8W 4AW..	Torsion....	359	270	0.99	1.70	4.98
25	1 year....	3W 9AD..	Compression	4 026	...	2.44	8.70	30.60
26	1 year....	3W 9AD..	Flexure.....	675	450	2.84	1.24	2.78
27	1 year....	3W 9AD..	Tension....	373	...	2.86	0.56	1.34
28	1 year....	3W 9AD..	Torsion....	476	357	1.23	2.15	5.13
29	1 year....	3W 9AW..	Compression	3 226	...	2.53	6.70	23.60
30	1 year....	3W 9AW..	Flexure.....	520	347	2.40	1.11	3.28
31	1 year....	3W 9AW..	Tension....	240	...	2.63	0.41	1.12
32	1 year....	3W 9AW..	Torsion....	290	217	0.95	1.58	3.99
33	1 year....	28W 11AD.	Compression	3 542	...	2.25	9.70	26.80
34	1 year....	28W 11AD.	Flexure.....	624	416	2.44	1.08	2.93
35	1 year....	28W 11AD.	Tension....	313	...	2.72	0.61	1.51
36	1 year....	28W 11AD.	Torsion....	469	352	1.33	1.98	4.95
37	1 year....	28W 11AW.	Compression	2 866	...	2.30	6.20	22.80
38	1 year....	28W 11AW.	Flexure.....	449	300	2.15	1.04	2.66
39	1 year....	28W 11AW.	Tension....	197	...	2.55	0.38	1.17
40	1 year....	28W 11AW.	Torsion....	323	242	1.02	1.50	3.50
41	1 year....	7W 12AD.	Compression	3 096	...	2.25	7.20	23.70
42	1 year....	7W 12AD.	Flexure.....	632	421	2.68	1.25	2.61
43	1 year....	7W 12AD.	Tension....	323	...	2.45	0.60	1.36
44	1 year....	7W 12AD.	Torsion....	431	323	1.21	1.77	3.71
45	1 year....	7W 12AW.	Compression	2 462	...	2.23	5.70	20.50
46	1 year....	7W 12AW.	Flexure.....	438	292	2.27	0.97	1.96
47	1 year....	7W 12AW.	Tension....	254	...	2.42	0.50	1.33
48	1 year....	7W 12AW.	Torsion....	330	247	1.09	1.49	3.69
49	1 year....	2W 12AD.	Compression	2 470	...	2.04	6.00	20.90
50	1 year....	2W 12AD.	Flexure.....	540	360	2.05	1.32	2.92
51	1 year....	2W 12AD.	Tension....	250	...	2.42	0.52	1.30
52	1 year....	2W 12AD.	Torsion....	416	312	1.11	1.77	3.84
53	1 year....	2W 12AW.	Compression	2 026	...	1.94	5.50	17.70
54	1 year....	2W 12AW.	Flexure.....	416	277	2.02	1.03	2.17
55	1 year....	2W 12AW.	Tension....	220	...	2.12	0.54	1.35
56	1 year....	2W 12AW.	Torsion....	300	225	1.10	1.34	3.09

* See Table 6 for details of curing and testing conditions.

TABLE 5.—(*Continued*)

Line No.	Total age (1)	Curing and testing condition* (2)	Kind of test (3)	Ultimate Strength, in Pounds per Square Inch: Approximate (4)	Ultimate Strength, in Pounds per Square Inch: Adjusted (by Upton's method) (5)	Modulus of elasticity, in millions of pounds per square inch (6)	Unit Deformation, in Ten Thousandths of an Inch per Inch, at: 50% of ultimate load (7)	Unit Deformation, in Ten Thousandths of an Inch per Inch, at: Ultimate load (8)
			(*b*) Series II*					
57	7 days...	Standard...	Compression	1 995	...	2.27	4.70	19.20
58	7 days...	Standard...	Flexure.....	527	351	2.51	1.02	2.30
59	7 days...	Standard...	Tension....	201	...	...	...	...
60	7 days...	Standard...	Torsion....	297	223	1.09	1.42	3.20
61	28 days...	Standard...	Compression	3 650	...	2.96	6.30	20.80
62	28 days...	Standard...	Flexure.....	571	381	3.21	0.91	2.10
63	28 days...	Standard...	Tension....	315	...	3.08	0.52	1.20
64	28 days...	Standard...	Torsion....	467	350	1.49	1.45	3.40
65	5 months.	Standard...	Compression	4 768	...	3.40	7.10	22.40
66	5 months.	Standard...	Flexure.....	707	471	3.75	0.96	2.40
67	5 months.	Standard...	Tension....	420	...	3.99	0.55	1.10
68	5 months.	Standard...	Torsion....	605	454	2.32	1.84	4.00

* See Table 6 for details of curing and testing conditions.

Flexural unit deformations are computed from the formula:

$$e = \frac{6\ y\ d}{L^2} \qquad (37)$$

in which, e = unit deformation in extreme fibers at mid-span; y = measured deflection at mid-span; d = depth of beam; and L = clear span. Torsional unit deformations are computed from the formula:

$$e_s = \frac{\rho_e\ \psi_L}{L_g} \qquad (38)$$

in which, e_s = unit linear shearing displacement at the extreme fiber; ρ_e = radius of the circular torsion specimen; ψ_L = measured total twist, in radians, for length, L_g; and, L_g = gauge length over which detrusion or twist is measured.

TABLE 6.—Descriptions of Ages, Curings, and Test Conditions

Item No.	Curing code number (1)	Curing Time: Moist (2)	Curing Time: Air of laboratory (3)	Curing Time: Total (4)	Test condition (5)
		(*a*) Series I			
1	Standard..	7 days..	....	7 days..	Wet
2	Standard..	28 days..	....	28 days..	Wet
3	Standard..	3 months	....	3 months	Wet
4	Standard..	1 year...	....	1 year...	Wet
5	8W 4AD..	8 months	4 months	1 year...	Dry
6	8W 4AW..	8 months	4 months	1 year...	Wet
7	3W 9AD..	3 months	9 months	1 year...	Dry
8	3W 9AW..	3 months	9 months	1 year...	Wet
9	28W 11AD	28 days..	11 months	1 year...	Dry
10	28W 11AW	28 days..	11 months	1 year...	Wet
11	7W 12A D	7 days...	12 months*	1 year....	Dry
12	7W 12AW	7 days...	12 months*	1 year....	Wet
13	2W 12A D	2 days...	12 months*	1 year....	Dry
14	2W 12AW	2 days...	12 months*	1 year....	Wet
		(*b*) Series II			
15	Standard.	7 days..	....	7 days..	Wet
16	Standard.	28 days..	....	28 days..	Wet
17	Standard.	5 months	....	5 months	Wet

* Slightly less than 12 months.

Table 6 supplements Table 5 by supplying the details concerning curings, ages, and test conditions. Items Nos. 1 to 4, inclusive, and Items Nos. 15, 16, and 17, were tested after undergoing the standard moist curing for the periods given in Table 6. The remainder were cured in the air of the laboratory for the stated periods, after being cured moist. Items Nos. 6, 8, 10, 12, and 14, however, differ from the corresponding test, Items Nos. 5, 7, 9, 11, and 13, in that they were immersed for 24 hr prior to the test.

In Series I (Table 6(*a*)) the concrete mix was 1: 3.25 by weight; the water-cement ratio was 1.00 (7.5 gal per sack of cement); the slump was 8 in; and the aggregate was crushed granite screenings of $\frac{3}{8}$-in. maximum size. Moist curing was by immersion after removal of the moulds at two days. Specimens exposed to air at the age of two days were immersed for about an hour immediately after the removal of the moulds (see Items Nos. 13 and 14, Table 6). It was unsafe to handle the slender beams prior to an age of two days and it was deemed desirable that all moulds be removed at the same age.

In Series II (Table 6(*b*)) the concrete mix was 1: $2\frac{1}{2}$: 2 by weight; the water-cement ratio was 1.00 (7.5 gal per sack of cement); the slump was 7 in.; the fine aggregate was sand from 0 to No. 4 mesh; and the coarse aggregate was pea gravel graded from $\frac{3}{8}$ in. to $\frac{1}{2}$ in.

The specimens for both series were of the same kind, as follows: Compressive specimens were standard 3 by 6-in. cylinders, parallel tests being made on 6 by 12-in. and 2 by 4-in. cylinders. Flexural specimens were 3 by 3 by 40-in. beams loaded at the center, simply supported and tested on a 38-in. span; tensile specimens were 3 by 12-in. cylinders, the ends of which were clamped in spherically seated grips; and torsional specimens were of the same type as the tensile specimens. Each compressive result in Table 5 is from at least five tests on duplicate specimens. Other test results are from two or more duplicate specimens. The agreement in results obtained from companion specimens was satisfactory.

Table 7 contains several types of ratios useful in appraising the results of the tests. For example, in Table 7(*a*), are listed the ratios of ultimate strengths in compression, flexure, and tension to those in torsion. This includes both the apparent flexural and torsional ultimates and the actual strengths, as determined by Upton's method. Table 7(*b*) gives similar ratios for stiffness, the property of which the modulus of elasticity is a measure.

Table 7(*c*) shows the relationship of maximum linear unit deformations in the extreme fibers in torsion to those in compression, flexure, and tension. Ratios are given for the ultimate deformations (that is, those corresponding to the maximum load carried) and for deformations corresponding to 50% of the maximum load carried. The relationship at 50% of the ultimate load is the more significant value since it lies virtually within the elastic range of stress for the mixtures of these series,[19] and is more dependable than ratios based on strains which were observed while the specimen was deforming rapidly and at the point of fracture. Nevertheless, the uniformity of the

[19] See plotted stress-strain diagrams for compression, flexure, tension, and torsion, Rept. of Committee, Arch Dam Investigation, Engineering Foundation, Vol. II pp. 476–477, 502–504, and 529–532.

TABLE 7.—COMPARISONS OF TORSIONAL PROPERTIES WITH THOSE IN COMPRESSION, FLEXURE, AND TENSION

Item No.*	(a) Ratios of Compressive, Flexural, and Tensile Strengths to the Torsion Strengths of Companion Specimens						(b) Ratios of Stiffness in Torsion to That in Compression, Flexure, and Tension †			(c) Ratios of Unit Linear Deformations in Torsion to Those in Compression, Flexure, and Tension					
	Nominal			Modified (Upton)						At 50% of Ultimate Load			At Ultimate Load		
	Compression torsion	Flexure torsion	Tension torsion	Compression torsion	Flexure torsion	Tension torsion	Torsion compression	Torsion flexure	Torsion tension	Torsion compression	Torsion flexure	Torsion tension	Torsion compression	Torsion flexure	Torsion tension
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
							(a) Series I								
1.....	6.28	1.42	0.83	8.36	1.26	1.10	0.53	0.36	0.46	0.30	2.00	4.40	0.22	1.58	4.65
2.....	7.75	1.38	0.67	10.39	1.23	0.89	0.38	0.34	0.37	0.32	2.01	4.38	0.24	1.95	4.57
3.....	9.50	1.46	0.74	12.70	1.30	0.98	0.46	0.35	0.42	0.20	1.73	3.39	0.23	2.32	4.88
4.....	9.65	1.25	0.66	12.85	1.12	0.89	0.48	0.52	0.52	0.22	1.60	3.05	0.17	1.68	3.04
5.....	7.97	1.33	0.76	10.65	1.19	1.02	0.54	0.50	0.47	0.21	1.47	2.70	0.14	1.50	2.52
6.....	10.37	2.01	0.78	13.75	1.78	1.03	0.37	0.35	0.40	0.23	1.33	3.33	0.19	1.75	3.20
7.....	8.45	1.42	0.78	11.28	1.26	1.04	0.51	0.44	0.43	0.25	1.73	3.85	0.17	1.84	3.82
8.....	11.10	1.79	0.83	14.90	1.60	1.11	0.38	0.40	0.36	0.24	1.42	3.88	0.17	1.21	3.54
9.....	7.55	1.33	0.67	10.05	1.18	0.89	0.59	0.55	0.49	0.20	1.82	3.24	0.19	1.68	3.27
10.....	8.90	1.39	0.61	11.90	1.24	0.82	0.44	0.48	0.40	0.24	1.44	3.95	0.15	1.31	2.98
11.....	7.20	1.47	0.75	9.60	1.30	1.00	0.54	0.45	0.49	0.25	1.42	2.95	0.16	1.88	2.73
12.....	7.48	1.33	0.77	10.00	1.18	1.03	0.49	0.48	0.45	0.26	1.53	2.98	0.18	1.88	2.76
13.....	5.95	1.30	0.60	7.95	1.15	0.80	0.54	0.54	0.46	0.30	1.34	3.40	0.18	1.31	2.94
14.....	6.78	1.39	0.73	9.02	1.23	0.98	0.57	0.55	0.52	0.24	1.30	2.48	0.17	1.42	2.29
							(b) Series II								
15.....	6.72	1.77	0.68	8.95	1.57	0.90	0.48	0.43	....	0.30	1.38	....	0.17	1.39	....
16.....	7.80	1.23	0.68	10.50	1.09	0.91	0.51	0.47	0.48	0.23	1.58	2.78	0.16	1.62	2.83
17.....	7.82	1.17	0.70	10.50	1.04	0.93	0.68	0.62	0.58	0.26	1.92	3.35	0.18	1.66	3.64

* For details of curing time and testing conditions, see corresponding item numbers in Table 6.

† That is, ratio of modulus of rigidity to modulus of elasticity, or Young's modulus: or $\frac{E'}{E}$.

ratios at the ultimate, and the excellent agreement of trend with those at 50%, is interesting.

Table 8 supplies comparisons of the age-strength relationship for the several types of loading for all specimens that have been "standard cured".

TABLE 8.—COMPARISON OF THE AGE-STRENGTH RELATIONSHIPS FOR STANDARD CURING (RATES OF INCREASE AS PERCENTAGES OF 28-DAY STRENGTH)

Item No.	Age	Compression		Flexure		Tension		Torsion	
		Strength, in pounds per square inch	Percentage of 28-day strength	Strength, in pounds per square inch	Percentage of 28-day strength	Strength, in pounds per square inch	Percentage of 28-day strength	Strength, in pounds per square inch	Percentage of 28-day strength
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
				(a) Series I					
1	7 days...	1 780	56	401	71	234	85	284	69
2	28 days...	3 200	100	570	100	275	100	413	100
3	3 months.	4 250	133	652	115	329	119	447	108
4	1 year....	4 850	152	631	111	333	121	504	122
				(b) Series II					
5	7 days...	1 995	55	527	93	201	64	297	64
6	28 days...	3 650	100	571	100	315	100	467	100
7	5 months.	4 768	131	707	123	420	133	605	127

Fig. 13 shows the torsional testing rig which was devised for these tests and the essential features of which were utilized for many subsequent torsional tests of the plaster-celite mixture used in the model of Boulder Dam and in Boulder Dam concrete researches conducted by the U. S. Bureau of Reclama-

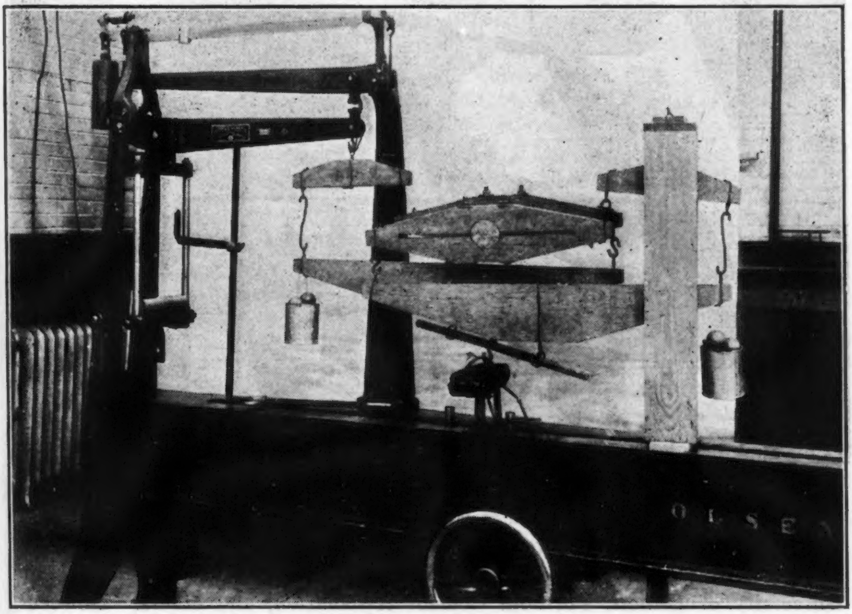

FIG. 13.—TORSIONAL TESTING RIG.

tion in its Denver, Colo., laboratories. In these tests it proved convenient to hook the rig on to the lever system of a hand-power beam testing machine which made it easy to apply known increments of load. Torque is applied entirely free of bending and excellent spiral fractures were obtained in all tests. The spiral fractures in these tests were longer than that shown in Fig. 8 of Mr. Andersen's paper. Any bending would tend to shorten the length of the fracture; or the difference may come from differences in the behavior of the respective concretes used. The angle of failure for Fig. 5 of the paper is more nearly similar to the breaks illustrated in Fig. 14.

The latter view shows some typical fractures and more detail of the rig and the troptometer which was used for measurements of detrusions. Two breaks per specimen were usually obtained. After the first break the troptometer could not be used because of the shortened gauge length, but the second break gave excellent checks on the ultimate torsional strength of the specimen. Apparently, the strength of the unbroken part of the specimen was not affected by the first test since second tests gave strengths that were sometimes slightly greater, and sometimes slightly less, than those at the first test.[20]

[20] For a complete drawing of the rig and troptometer, see p. 455, of the Engineering Foundation, Arch Dam Investigation, Vol. II.

FIG. 14.—DETAIL OF TROPTOMETER, AND TYPICAL TORSION FAILURES.

Fig. 15 shows the three sizes of compressive specimens fully harnessed with longitudinal compressometers and lateral extensometers.[21] The compressometers were similar to those in use for years at the laboratories of the University of Illinois, Portland Cement Association, and others. However, their use on specimens as small as 3 by 6 in. and 2 by 4 in. had not been attempted elsewhere. The lateral extensometers were improvised for these tests and similar ones have since been devised by various laboratories.

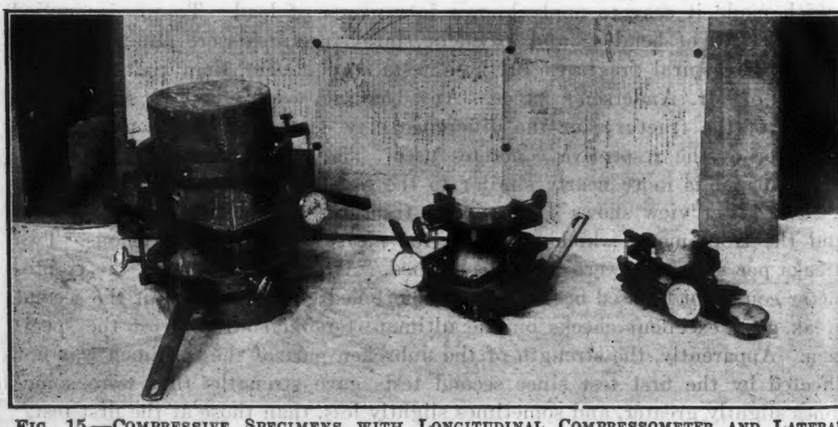

FIG. 15.—COMPRESSIVE SPECIMENS WITH LONGITUDINAL COMPRESSOMETER AND LATERAL EXTENSOMETERS.

[21] See, also, pp. 467 to 469, and pp. 472 and 473, of the Engineering Foundation, Arch Dam Investigation, Vol. II.

Fig. 16 shows the tensile grips, the tensile extensometer (the one with the "last-word" dials in the background), and a number of representative tensile breaks. In the foreground, and at the extreme right, are shown parts of another extensometer[22] not used in these tests. The tensile extensometers were

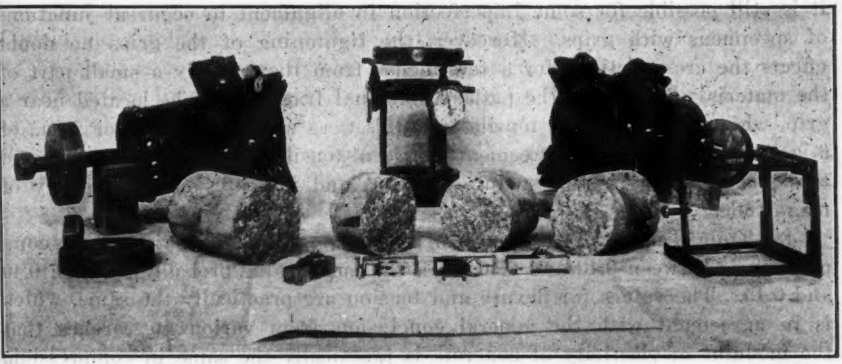

FIG. 16.—TENSILE GRIPS AND TENSILE EXTENSOMETER, WITH A NUMBER OF REPRESENTATIVE TENSILE BREAKS.

developed for these tests and the tensile grips were patterned after grips previously used by the Portland Cement Association. After one test, with stress-strain data taken, several additional breaks with only the ultimate load observed could be obtained in the tensile tests. The strengths agreed well with those from the first break, as was true, also, for torsion.

A study of Tables 7 and 8 appears to justify the following generalizations within the scope of these tests:

1.—From Table 7(*a*) it is evident that the apparent torsional ultimate strengths (Equation (36) for M_t at ultimate) can be expected to be from one-sixth to one-ninth the ultimate compressive strength, with one-eighth (say, $12\frac{1}{2}\%$) a fair average value. The true or modified torsional strength, as corrected by Upton's method, $v = \frac{3}{4}\frac{M_t\,\rho_e}{J}$ (see Table 5 and text description relating to it), will be about one-sixth, or 17%, of the ultimate strength in compression.

2.—The apparent ultimate strength in torsion is about 70% of the apparent ultimate strength in flexure (Equation (35)), and the true torsional strength is about 80% of the true flexural strength $\left(f = \frac{2}{3}\frac{M\,c}{I}\right)$, as obtained by Upton's method.

3.—The apparent ultimate strength in torsion is about 25% greater than the observed tensile strength, but the true torsional strength is the same as the

[22] See, also, pp. 452–453, 456–457, Engineering Foundation, Arch Dam Investigation, Vol. II.

ultimate tensile strength as determined by test. The evidence on this score strongly supports Conclusion 2 of the paper. When bending is eliminated as fully as it was in the writer's tests and probably also in Mr. Andersen's torsional tests, it is possible that the torsion test gives the better measure of tensile strength. While the tensile grips themselves were spherically seated, it is still possible for some imperfection in alignment to occur at junctions of specimens with grips. Moreover, the tightening of the grips no doubt effects the cross-sections for a few inches from them. Only a small part of the material that lies in the path of torsional fracture can be located near a grip, whereas many of the tensile fractures were within the 1 in., or 2 in., of a grip. The excellent agreement between tensile strengths and corrected torsional strengths is re-assuring, however, and indicates no discrepancy of magnitude.

4.—From Table 7(*b*) the ratio of torsional stiffness to stiffness in compression is between 0.40 and 0.50, with the mean value probably between 0.40 and 0.45. The values for flexure and tension are practically the same, which is in agreement with the general conclusion from various researches that the modulus of elasticity of concrete is practically the same in compression, flexure, and tension. For values of Poisson's ratio between 0.15 and 0.25, E' would lie between 0.435 E and 0.40 E, according to the formula,

$$E' = \frac{E}{2(1 + \sigma)} \qquad \text{.......................(39)}$$

in which, E' = the modulus of elasticity in shear (modulus of rigidity); E = modulus of elasticity; and σ = Poisson's ratio.[23]

Poisson's ratio was determined for all specimens tested in compression and was found to lie between 0.15 and 0.25, with about 0.20 as a fair average.[24] There apears to be no consistent variation in Poisson's ratio for any known variable (strength, richness of mixture, curing, or test condition, etc.) except a slight tendency to increase with the age at test under standard curing. This statement is based upon many test results from many different series and mixtures. The extreme range is from about 0.10 to slightly more than 0.30, the higher values having been from specimens several years old at test.

5.—From Table 7(*c*) the torsional linear unit displacement at the extreme fibers for 50% of the ultimate load is about 0.25 that in compression, 1.5 that in flexure, and more than three times that in tension. The ratios at ultimate load differ little from these values. In noting the fact that maximum torsional deformations exceed maximum tensile deformations in a ratio of virtually 3 to 1, it should be kept in mind that the average torsional deformation is much less than the maximum. The same thought applies likewise to the beam which finally fails progressively in tension, as does the torsional specimen.

6.—Table 5 shows wide variations in strengths, moduli of elasticities and deformations for the different ages, curings, and test conditions. It is ex-

[23] See any textbook on Strength of Materials, such as that by James E. Boyd, Third Edition (1924), p. 58.

[24] Engineering Foundation, Arch Dam Investigation, Vol. II, pp. 476, 477, 487, 529, and 536.

tremely satisfying, therefore, to note that not one of these major variables appears to have any consistent effect upon the ratio of the torsional (or other) property to the corresponding property in compression, flexure, or tension. In other words, variations in mixture, materials, age, curing, and test condition effect the compressive, flexural, tensile, and torsional properties in essentially the same manner and to the same relative extent. (Support for the author's Conclusion 1 follows as a corollary of the writer's Conclusion 6.)

7.—Data as to the relationship of age to strength for the standard curing condition are brought together in Table 8, in which the rates of increase of strengths with ages are expressed as percentages of the respective 28-day strengths. In general, the torsional behavior is very similar to that in flexure and tension. In all three of these respects, a higher percentage of the ultimate strength is developed in the early stages of the curing period than for compression. Not all the data for Series II support this statement, but other tests bear it out as far as tension and flexure are concerned.[25]

In this discussion no mention has been made of the relative toughness and resilience of the concretes in compression, flexure, tension, and torsion. The areas under the stress-strain diagrams are taken as measures of this property.[26]

Acknowledgment.—The varied data of this discussion were obtainable only because of the hearty support that was accorded the auxiliary test program by J. L. Savage and Ivan E. Houk, Members, Am. Soc. C. E., of the U. S. Bureau of Reclamation Staff, functioning under R. F. Walter, M. Am. Soc. C. E., Chief Engineer, and Elwood Mead, M. Am. Soc. C. E., Commissioner of Reclamation.

A. A. EREMIN,[27] ASSOC. M. AM. SOC. C. E. (by letter).[27a]—A new method of computing the shearing stresses distributed in plain concrete and reinforced concrete members sustaining torsion, is developed in this paper. In his tests the author studied the characteristics of members with square and round sections. In bridges and buildings, however, it occurs more often that the members sustaining torsion are of rectangular section. Torsion shearing stresses distributed in rectangular sections of reinforced concrete members may be analyzed by means of Equation (13) or Equation (14), assuming an equivalent round section.

The maximum torsion shearing stress along the center, of the wide side of a rectangular section of a plain concrete member, as developed by Professor Bach,[28] is,

$$v_m = \left(3 + \frac{2.6}{0.45 + \dfrac{B}{a}}\right) \frac{M}{8\,B\,a^2} \quad \text{.............. (40)}$$

[25] Strength-age graphs are shown for the same tests on pp. 461 and 526 of the Engineering Foundation, Arch Dam Investigation, Vol. II. On pp. 525 and 526, the torsional strength at 5 months should be 605 instead of 570. For the various curings and test conditions plotted graphs can be found on pp. 429, 438, 463, 464, and 465.

[26] Graphic comparisons are shown on pp. 496, 497, 539, and 540 of the Engineering Foundation, Arch Dam Investigation, Vol. II.

[27] Assoc. Bridge Designing Engr., Bridge Dept., Div. of State Highways, California Public Works, Sacramento, Calif.

[27a] Received by the Secretary December 8, 1934.

[28] "Deutscher Auschuss für Eisenbeton", Heft 16, 1912, W. Ernst & Sohn, Berlin, Germany.

in which, $2B$ = depth of the wide face; and $2a$ = width of the narrow face. From Equations (11) and (40), the radius of an equivalent round section may be determined, as follows:

$$\rho_e = \sqrt[3]{\frac{16\,B\,a^2}{\pi\left(3 + \dfrac{2.6}{0.45 + \dfrac{B}{a}}\right)}} \quad \ldots\ldots\ldots\ldots\ldots (41)$$

In a square section, $2B = 2a$; therefore, from Equation (41), the radius of the equivalent round section for a square is $\rho_e = 1.02\,B$, or, approximately, a circle inscribed in the square, as shown by the author.

The efficiency coefficient for steel reinforcement at the wide and narrow faces of a rectangular member may be computed by Equation (15). The efficiency coefficient of steel along the narrow side of a rectangular section must be reduced by the ratio of the maximum shearing stress along the narrow side to the maximum shearing stress along the wide side, r.

The maximum torsional shearing stress along the center of the narrow side of a rectanguluar section is,

$$v_m = \left(3 + \frac{2.6}{0.45 + \dfrac{B}{a}}\right)\frac{M}{8\,a\,B^2} \quad \ldots\ldots\ldots\ldots\ldots (42)$$

Therefore, from Equations (40) and (42), the ratio, r, is,

$$r = \frac{a}{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

Equations (40) and (42), for the maximum torsional shearing stresses, were derived by Professor Bach from tests of concrete members with rectangular sections for which the ratio, r, varied from square section proportions to $r = 0.5$. For more conclusive results further theoretical and experimental study is needed, of shear distribution due to torsion in the wider rectangular sections.